ON THE MODULI STACK OF COMMUTATIVE, 1-PARAMETER FORMAL LIE GROUPS

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ABSTRACT. We commence a general algebro-geometric study of the moduli stack of commutative, 1-parameter formal Lie groups. We emphasize the pro-algebraic structure of this stack: it is the inverse limit, over varying n, of moduli stacks of n-buds, and these latter stacks are algebraic. Our main results pertain to various aspects of the height stratification relative to fixed prime p on the stacks of buds and formal Lie groups.

INTRODUCTION

The aim of this paper is to explicate some of the basic algebraic geometry of the moduli stack of commutative, 1-parameter formal Lie groups. Formal group laws have received extensive study in the mathematical literature for well over 50 years. Their applications cut across a wide swath of mathematics; Hazewinkel's massive treatise [Ha] of almost 30 years ago already counted over 500 citations. We shall make no attempt to survey the vast literature produced since then. But we do wish to signal one area of mathematics in which the moduli stack of formal groups has appeared in an explicit and important way: namely, *stable homotopy theory*.

Formal group laws enter into topology in the study of generalized cohomology theories: to any even, periodic cohomology theory on the category of topological spaces, one may associate a formal group law in a way canonical up to choice of coordinate; see, for example, Lurie's survey article [Lur]. In [Q], Quillen observed that the formal group law associated to complex cobordism MU is *universal*. In fact, the affine schemes $\operatorname{Spec} MU_*$ and $\operatorname{Spec} MU_*MU$ admit a natural internal groupoid structure

$$(*) \qquad \operatorname{Spec} MU_*MU \rightrightarrows \operatorname{Spec} MU_*$$

which represents the functor

$$(\text{Affine schemes})^{\mathrm{opp}} \longrightarrow \text{Groupoids}$$
$$\operatorname{Spec} R \longmapsto \left\{ \begin{array}{c} \text{formal group laws and} \\ \text{strict isomorphisms over } R \end{array} \right\}.$$

Thus the stack of formal groups arises *naturally* in topology.

In the 1970's, Morava began to emphasize the role of group laws of given *height*, and in particular the *Morava stabilizer groups*, in stable homotopy theory, and he advocated for the importation of algebraic geometry into the subject as a means to gain conceptual insight. The notion of height plays a fundamental role in homotopy theory's so-called *chromatic picture*; see Ravenel's important reference [R1]

for details. More recently, owing much to the influence and insight of Hopkins, topologists have come to understand that a great deal of the chromatic picture's impressive computational architecture is, in some sense, governed by the structure of the moduli stack of formal Lie groups; see the introduction to [GHMR] for some discussion.

For example, in [Hop, 21.4], Hopkins interprets the Landweber exact functor theorem as a statement closely related to flatness of quasi-coherent sheaves on the stack of formal groups. In [P, 4.8], Pribble obtains a purely stack-theoretic analog of the chromatic convergence theorem of Hopkins-Ravenel [R2, §8.6]. In [N, 26], Naumann gives a satisfying algebro-geometric explanation of Hovey's and Strickland's result [HS, 4.2] that for any two Landweber-exact BP_*-algebras R and S of the same height[1], the comodule categories of the Hopf algebroids

$$(R, R \otimes_{BP_*} BP_*BP \otimes_{BP_*} R) \quad \text{and} \quad (S, S \otimes_{BP_*} BP_*BP \otimes_{BP_*} S)$$

are equivalent: namely, he shows that the underlying *stacks* are equivalent. Many important change of rings theorems in stable homotopy theory can be seen as arising in this way.

Despite the apparent importance and utility of the moduli stack of formal Lie groups in homotopy theory (to say nothing of whatever applications it may have in other branches of mathematics), surprisingly little foundational material on this stack has yet appeared in the mathematical literature. Our intent in this paper is to take some of the first steps towards filling this gap. That said, let us hasten to add that many things we shall discuss have already been treated elsewhere in one form or another. Notably, [N, §6] contains a quick account of some of the basic algebro-geometric moduli theory of formal groups: in particular, Naumann gives an algebro-geometric definition of formal groups, shows that they form an fpqc-stack, and gives an intrinsic description of the stack associated to (∗) as the stack of formal groups with trivialized canonical bundle. The *height stratification* is defined and plays a prominent role in [P, §4.4]. Hopkins has covered a considerable amount of moduli theory in [Hop] and in other courses at MIT. And there have been many papers — let us signal especially [L] — devoted to classification of formal group *laws*; it is this work, of course, on which ours shall ultimately rest.

The essential feature of this paper is, perhaps then, its *scope*. We have tried to develop the moduli theory largely from the ground up, beginning from the foundations of the classical formal group law literature. Although the topologists have emerged as the primary consumers of the theory, let us now extend our apologies to them: we have aproached the subject as a pure piece of algebraic geometry, with essentially no regard for considerations arising in topology aside from a few remarks in (4.3.6). For a much more comprehensive account of the moduli stack of formal Lie groups and its relation to stable homotopy theory, we refer to the forthcoming [Go2].

Let us now discuss the main contents of this paper.

Section 1 serves chiefly to collect terminology surrounding the various objects at play. Following, for example, [G, Me], we shall refer to the algebro-geometric version of a formal group law as a "formal Lie group", as opposed to just "formal group". Though the "Lie" is now less commonly used than in earlier treatments of the

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[1]This notion of height is closely related to, but in some sense is not completely compatible with, the notion of height we use when speaking of the *height stratification*; see (4.3.6).

subject, we feel that "formal group" should be reserved for any group object arising in a formal-geometric context. Our choice of terminology (re)emphasizes that the objects we study are the algebro-geometric analog of an infinitesimal neighborhood of the identity of a (1-dimensional) Lie group; see (1.2.1) and (1.3.1).

In §2 we address the first properties of the stack of formal Lie groups $\mathscr{FLG}$ and of the related stacks we consider. Unfortunately, $\mathscr{FLG}$ is not *algebraic* in the sense traditional in algebraic geometry [LMB, 4.1]: for example, its diagonal is not of finite type [LMB, 4.2]. One remedy for this defect, due to Hopkins and followed in [Go1, 1.8; P, 3.15; N, 6], is the following. Let Λ denote the Lazard ring. Then the universal formal group law specifies a map

$$(**) \qquad\qquad \operatorname{Spec} \Lambda \longrightarrow \mathscr{FLG}$$

which is at least surjective, flat, and affine (it is not an honest presentation because it is not of finite type). So one simply *redefines* the notion of algebraic stack to mean an "affine-ized" version of the usual one, using flat covers, so that $(**)$ is a presentation and $\mathscr{FLG}$ is algebraic. See [Hol] for an axiomatization of the idea.

This modified definition carries several advantages. One is that a considerable amount of algebraic geometry may still be done "as usual" on such stacks. Amongst examples of import to the topologists, we note that there are satisfactory notions of closed substack, quasi-coherent sheaf, etc. Another advantage is that the 2-category of Hopf algebroids becomes antiequivalent to the 2-category of algebraic stacks equipped with a presentation, and the category of comodules for a given Hopf algebroid becomes equivalent to the category of quasi-coherent sheaves on the associated stack. Hence a useful link is forged to homotopy theory.

On the other hand, the modified definition is ultimately awkward from the point of view of geometry: many examples of objects that ought to be algebraic are not[2], including even all non-quasi-compact *schemes*. We shall view the modified definition as unsuitable. But we are still left with the problem of finding a good setting in which to regard $\mathscr{FLG}$ as a geometric object. One possibility is to again attempt to weaken the notion of algebraic stack, starting, say, by requiring our presentations only to be fpqc coverings by schemes. Or, at least in the case of $\mathscr{FLG}$, we could get by with presentations that are a bit more specialized; see [Go1, 1.2, 1.13] for some musings on this point. A satisfactory approach along these lines would certainly be highly desirable. But many phenomena in algebraic geometry are only well understood with certain finiteness hypotheses in place, and we believe that such an approach is likely to present some untoward foundational issues, at least weighed against the immediate aims of this paper. So we shall take another tack.

We shall continue to understand algebraic stack in the traditional sense, as in [LMB]. Then, indeed, $\mathscr{FLG}$ is not algebraic. But in some sense it is not far from algebraic. Namely, we may naturally describe it as a *pro-algebraic* stack. We do so by considering the algebro-geometric classification of what are classically known as *n-bud laws* (1.1.2). These were introduced in Lazard's seminal paper [L]; informally, they are just truncated formal group laws. The moduli stack $\mathscr{B}_n$ of n-buds is a perfectly good algebraic stack (2.3.2), and, quite as a formal group law is a "limit"

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[2]A phenomenon already present in examples of interest to homotopy theorists, as noted in [Go1, footnote 5].

of n-bud laws, we obtain the stack of formal Lie groups as the "limit" of the stacks $\mathscr{B}_n$ (2.6.8).

In §§3 and 4 we turn to the essential feature of the geometry of the stacks $\mathscr{B}_n$ and $\mathscr{FLG}$, respectively, namely the *height stratification* relative to a fixed prime p. These sections form the core of the paper. The height stratification on $\mathscr{FLG}$ consists of an infinite descending chain of closed substacks

$$\mathscr{FLG} = \mathscr{FLG}^{\geq 0} \supsetneq \mathscr{FLG}^{\geq 1} \supsetneq \cdots,$$

and, for each n, the height stratification on $\mathscr{B}_n$ consists of a finite descending chain of closed substacks

$$\mathscr{B}_n = \mathscr{B}_n^{\geq 0} \supsetneq \mathscr{B}_n^{\geq 1} \supsetneq \cdots.$$

As n varies, the stratifications on $\mathscr{B}_n$ are compatible in a suitable sense, and their "limit" recovers the stratification on $\mathscr{FLG}$ (4.2.2).

One of our main results in the following.

Theorem (3.5.7). *$\mathscr{B}_n$ is smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension -1 at every point, and, when it is defined, $\mathscr{B}_n^{\geq h}$ is smooth over $\operatorname{Spec}\mathbb{F}_p$ of relative dimension $-h$ at every point.*

We emphasize that the theorem asserts not only that the locally closed strata are smooth, but that the closed subschemes *defining* the stratification are themselves smooth. Hence each stratum has smooth closure.

Much of our subsequent effort is devoted to studying the strata $\mathscr{FLG}^h \subset \mathscr{FLG}$ and $\mathscr{B}_n^h \subset \mathscr{B}_n$ of height h formal Lie groups and n-buds, respecitvely. It is a classical result of Lazard [L, Théorème IV] that over a separably closed field of characteristic p, formal group laws are classified up to isomorphism by their height. We obtain a generalization as follows. Let $H = H_h$ be a "Honda" formal group law of height h defined over $\mathbb{F}_p$; see (3.1.16). Then for any $\mathbb{F}_p$-scheme S, we may view H as a group law over $\Gamma(S, \mathscr{O}_S)$, and we define the functor $\mathscr{A}ut(H)\colon S \mapsto \operatorname{Aut}_{\Gamma(S,\mathscr{O}_S)}(H)$.

Theorem (4.3.8). *$\mathscr{FLG}^h$ is equivalent to the classifying stack $B\big(\mathscr{A}ut(H)\big)$ for the fpqc topology.*

There is a version of the theorem when working with the stack $\mathscr{FLG}_{\mathrm{tr}}^h$ of formal Lie groups of height h with trivialized canonical bundle. One has an exact sequence

$$1 \longrightarrow \mathscr{A}ut_{\mathrm{str}}(H) \longrightarrow \mathscr{A}ut(H) \longrightarrow \mathbb{G}_m,$$

where $\mathscr{A}ut_{\mathrm{str}}(H)$ is the sub-group functor of *strict* automorphisms of H. Then one obtains $\mathscr{FLG}_{\mathrm{tr}}^h \approx \mathscr{A}ut(H)\backslash\mathbb{G}_m$. Of course, here $\mathbb{G}_m$ acts naturally on the right-hand side; the action appears on the left-hand side as $\mathbb{G}_m$'s natural action on trivializations, and this action realizes the forgetful functor $\mathscr{FLG}_{\mathrm{tr}}^h \to \mathscr{FLG}^h$ as a $\mathbb{G}_m$-torsor. We note that when K is a field containing $\mathbb{F}_{p^h}$, $\mathscr{A}ut(H)(K)$ is precisely the hth Morava stabilizer group studied in homotopy theory.

There is also a version of (4.3.8) for buds. Let $H^{(n)}$ denote the n-bud law obtained from H by discarding terms of degree $\geq n+1$, and let $\mathscr{A}ut(H^{(n)})$ denote the functor sending each $\mathbb{F}_p$-scheme $S \mapsto \operatorname{Aut}_{\Gamma(S,\mathscr{O}_S)}(H^{(n)})$.

Theorem (3.6.8). *When it is defined, $\mathscr{B}_n^h$ is equivalent to the classifying stack $B\big(\mathscr{A}ut(H^{(n)})\big)$ for the finite étale topology* [SGA3$_\mathrm{I}$, IV 6.3].

The results (3.6.8) and (4.3.8) accord the groups $\mathscr{A}ut(H^{(n)})$ and $\mathscr{A}ut(H)$ important places in the theory. We investigate their structure in the following way. For each n, there is a natural filtration of normal subgroups

$$\mathscr{A}ut(H^{(n)}) =: \mathscr{A}_0^{H^{(n)}} \supset \mathscr{A}_1^{H^{(n)}} \supset \cdots \supset \mathscr{A}_{n-1}^{H^{(n)}} \supset \mathscr{A}_n^{H^{(n)}} := 1$$

and an infinite filtration of normal subgroups

$$\mathscr{A}ut(H) := \mathscr{A}_0^H \supset \mathscr{A}_1^H \supset \mathscr{A}_2^H \supset \cdots ;$$

see (3.7.2) and (4.4.4), respectively. In the case of $\mathscr{A}ut(H)$, the $\mathscr{A}_\bullet^H$-topology recovers the usual topology on the Morava stabilizer group.

We calculate the successive quotients of the $\mathscr{A}_\bullet^{H^{(n)}}$- and $\mathscr{A}_\bullet^H$-filtrations as follows. For the $\mathscr{A}_\bullet^{H^{(n)}}$-quotients, let l be the nonnegative integer such that $p^l \leq n < p^{l+1}$, and assume $h \leq l$.

Theorem (3.7.3, 4.4.7). *There are natural identifications of presheaves on* $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}} \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ldots,\ p^{l-h}-1; \\ \mathbb{G}_a, & i = p^{l-h+1}-1,\ p^{l-h+2}-1,\ldots,\ p^l-1; \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\mathscr{A}_i^H/\mathscr{A}_{i+1}^H \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ p^3-1,\ldots ; \\ 0, & \text{otherwise}. \end{cases}$$

Here $\mu_{p^h-1} \subset \mathbb{G}_m$ is the sub-group scheme of (p^h-1)th roots of unity, and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}} \subset \mathbb{G}_a$ is the sub-group scheme of fixed points for the p^hth-power Frobenius operator. It follows that $\mathscr{A}ut(H^{(n)})$ is a smooth group scheme over $\mathbb{F}_p$ of dimension h (3.7.4).

In addition to (4.3.8), we obtain another description of $\mathscr{F}\mathscr{L}\mathscr{G}^h$ via a classical theorem of Dieudonné [D, Théorème 3] and Lubin [Lub, 5.1.3], the full details of which we shall not recall here. Very roughly, their theorem characterizes $\mathrm{Aut}_{\mathbb{F}_{p^h}}(H)$ as the profinite group G of units in a certain p-adic division algebra; see (4.4.10) for a precise formulation.

Theorem (4.6.2). *There is an equivalence of stacks over* $\mathbb{F}_{p^h}$,

$$\mathscr{F}\mathscr{L}\mathscr{G}^h \times_{\mathrm{Spec}\,\mathbb{F}_{p^h}} \mathrm{Spec}\,\mathbb{F}_{p^h} \approx \varprojlim B(G/N),$$

where the limit is taken over all open normal subgroups N *of* G.

The theorem is really a corollary of Dieudonné's and Lubin's theorem and of (4.6.1), where we show that $\mathscr{F}\mathscr{L}\mathscr{G}^h$ is a limit of certain classifying stacks of *finite étale* (but nonconstant) groups over $\mathbb{F}_p$. These groups all become constant after base change to $\mathbb{F}_{p^h}$.

In §5 we describe some aspects of the stacks $\mathscr{F}\mathscr{L}\mathscr{G}$ and $\mathscr{B}_n$ related to separatedness and properness.

Theorem (5.1). $\mathscr{B}_n$ *is universally closed over* $\mathrm{Spec}\,\mathbb{Z}$, *and, when it is defined,* $\mathscr{B}_n^{\geq h}$ *is universally closed over* $\mathrm{Spec}\,\mathbb{F}_p$.

The stacks $\mathscr{B}_n$ and $\mathscr{B}_n^{\geq h}$ fail to be proper because they are not separated; see (5.2). The failure of separatedness prevents us from concluding in a formal way that $\mathscr{F\!L\!G}$ and $\mathscr{F\!L\!G}^{\geq h}$ also satisfy the valuative criterion of universal closedness. Nevertheless, these stacks do satisfy the valuative criterion in many cases; see (5.4). By contrast, we deduce from our theory that the *stratum* $\mathscr{F\!L\!G}^h$ does satisfy the valuative criterion of separatedness.

Theorem (5.5). *Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Then $\mathscr{F\!L\!G}^h(\mathscr{O}) \to \mathscr{F\!L\!G}^h(K)$ is fully faithful for all $h \geq 1$.*

When $\mathscr{O}$ is a discrete valuation ring, the theorem is a (very) special case of de Jong's general theorem [dJ, 1.2] that restriction of p-divisible groups from $\operatorname{Spec} \mathscr{O}$ to the generic point $\operatorname{Spec} K$ is fully faithful.

We have included an appendix at the end describing some of the basic theory of limits in 2-categorical contexts; or more precisely, of limits in bicategories.

It is our hope that our study of the moduli stack of formal Lie groups will serve as a rough model for treating the related stack of p-divisible groups for fixed prime p. Indeed, this last stack also admits a pro-algebraic description: it is a limit, over varying n, of algebraic stacks of certain finite locally free group schemes of order p^n.

This paper is a condensed and slightly reorganized version of the author's Ph.D. thesis [Sm].

Notation and conventions. Except where noted otherwise, we adopt the following notation and conventions.

We relate objects in a category by writing

- $=$ for equal;
- $\cong$ for canonically isomorphic;
- $\simeq$ for isomorphic; and
- $\approx$ for equivalent or 2-isomorphic (e.g. for categories, fibered categories, stacks, etc.).

For simplicty, we often label arrows obtained in a tautological or canonical way by $\xrightarrow{\text{can}}$. Context will always make the precise meaning clear.

We write (Sets) for the category of sets, (Gp) for the category of groups, and (Sch) for the category of schemes. We write $\operatorname{Fib}(\mathscr{C})$ for the 2-category of fibered categories over a category $\mathscr{C}$. We abbreviate the term "category fibered in groupoids" by CFG, and we write $\operatorname{CFG}(\mathscr{C})$ for the 2-category of CFG's over a category $\mathscr{C}$. By a *stack*, we mean any fibered category in which morphisms and objects satisfy descent. In particular, we do not require stacks to be CFG's, contrary to many authors' convention; compare, for example, [LMB, 3.3.1].

Given an object S in a category $\mathscr{C}$, we write $\mathscr{C}_{/S}$ for the overcategory of S-objects, that is, of objects T equipped with a morphism $T \to S$ in $\mathscr{C}$. When $\mathscr{C} = (\mathrm{Sch})$ and S is an affine scheme $\mathrm{Spec}\,A$, we often write $(\mathrm{Sch})_{/A}$ instead of $(\mathrm{Sch})_{/S}$.

Given an object S in a category $\mathscr{C}$, we typically write just S again for both the presheaf of sets on, and fibered category over, $\mathscr{C}$ determined by S, $T \mapsto \mathrm{Hom}_{\mathscr{C}}(T, S)$. When clarity demands that the notation distinguish between S and its associated presheaf or fibered category, we denote the latter two by $\underline{S}$.

By default, "sheaf" or "presheaf" means "sheaf of sets" or "presheaf of sets", respectively.

When working in a 2-categorical context (for example, the case of fibered categories and stacks), we say that a diagram of objects and 1-morphisms

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

is *Cartesian* if the two composites $A \to D$ are isomorphic, and the choice of an isomorphism induces an equivalence $A \xrightarrow{\approx} C \times_D B$.

All rings are commutative with 1. We write $k(s)$ for the residue field at the point s of a scheme. Given a ring A and an A-module M, we write $\widetilde{M}$ for the quasi-coherent $\mathscr{O}_{\mathrm{Spec}\,A}$-module on $\mathrm{Spec}\,A$ obtained from M.

Given a scheme S, we write $\Gamma(S)$ for the global sections $\Gamma(S, \mathscr{O}_S)$ of the structure sheaf. In addition, for each integer $n \geq 0$ and indeterminates $T_1, \ldots, T_m$, we define the ring

$$
\begin{aligned}
\Gamma_n(S; T_1, \ldots, T_m) &:= \Gamma\big(S, \mathscr{O}_S[T_1, \ldots, T_m]/(T_1, \ldots, T_m)^{n+1}\big) \\
&\cong \Gamma(S)[T_1, \ldots, T_m](T_1, \ldots, T_m)^{n+1},
\end{aligned}
$$

where $\mathscr{O}_S[T_1, \ldots, T_m]/(T_1, \ldots, T_m)^{n+1}$ is the sheaf on S defined on each open subset U by

$$
U \longmapsto \Gamma(U, \mathscr{O}_S)[T_1, \ldots, T_m]/(T_1, \ldots, T_m)^{n+1}.
$$

In particular, we have $\Gamma_0(S; T_1, \ldots, T_m) \cong \Gamma(S)$.

1. Definitions

We begin by introducing some of the basic language and notation related to the objects we study in this paper.

1.1. **Review of formal group laws I.** Before beginning to study formal group laws from the point of view of algebraic geometry, it will be convenient to briefly review some of the foundations of the classical algebraic theory. We will review aspects of the classical theory related to the notion of *height*, as well as some refinements to the material we discuss here, in §3.1. The reader may wish to skip this section and refer back only as needed.

Let A be a ring.

Definition 1.1.1. A *formal Lie group law*, or just *formal group law, over A* is a power series $F(T_1, T_2) \in A[[T_1, T_2]]$ in two variables satisfying

(I) (identity) $F(T, 0) = F(0, T) = T$;
(A) (associativity) $F\big(F(T_1, T_2), T_3\big) = F\big(T_1, F(T_2, T_3)\big)$; and

(C) (commutativity) $F(T_1, T_2) = F(T_2, T_1)$;

here the equalities are of elements in the rings $A[[T]]$, $A[[T_1, T_2, T_3]]$, and $A[[T_1, T_2]]$, respectively. A *homomorphism* $F \to G$ of formal group laws over A is a power series $f(T) \in A[[T]]$ with constant term 0 satisfying

$$(*) \qquad\qquad f\bigl(F(T_1, T_2)\bigr) = G\bigl(f(T_1), f(T_2)\bigr)$$

in $A[[T_1, T_2]]$.

One obtains the notion of a bud law as a "truncated" version of (1.1.1). Let $n \geq 0$.

Definition 1.1.2. An *n-bud law over A* is an element

$$F(T_1, T_2) \in A[T_1, T_2]/(T_1, T_2)^{n+1}$$

satisfying conditions (I), (A), and (C) of (1.1.1) in the rings

$$A[T]/(T)^{n+1}, \ \ A[T_1, T_2, T_3]/(T_1, T_2, T_3)^{n+1}, \ \text{and} \ A[T_1, T_2]/(T_1, T_2)^{n+1},$$

respectively. A *homomorphism* $F \to G$ of n-bud laws over A is an element $f(T) \in A[T]/(T)^{n+1}$ with constant term 0 satisfying $(*)$ in the ring $A[T_1, T_2]/(T_1, T_2)^{n+1}$.

Example 1.1.3. Over any ring,
- the *additive law* or *n-bud* is given by $F(T_1, T_2) = T_1 + T_2$; and
- the *multiplicative law* or *n-bud*, $n \geq 2$, is given by

$$F(T_1, T_2) = (1 + T_1)(1 + T_2) - 1 = T_1 + T_2 + T_1 T_2.$$

Remark 1.1.4. We recall that any group law or bud law F over A is automatically equipped with a unique inverse homomorphism $i(T) = i_F(T) \colon F \to F$ satisfying $F\bigl(i(T), T\bigr) = F\bigl(T, i(T)\bigr) = 0$.

Remark 1.1.5. Let F be a group law or bud law over A. Then the axioms endow the monoid $\operatorname{End}_A(F)$ with a natural (noncommutative) *ring* structure: multiplication is composition of power series in the group law case, or of truncated polynomials in the bud law case, and addition is application of F; that is, the sum of $f(T)$ and $g(T)$ is $F\bigl(f(T), g(T)\bigr)$. We denote the sum in $\operatorname{End}_A(F)$ by $f +_F g$ to avoid confusion with usual addition of power series or of truncated polynomials. The unit element in $\operatorname{End}_A(F)$ is just $\operatorname{id}(T) := T$.

Remark 1.1.6. It is convenient to now make the following technical, though obvious, remark, which we'll use in §3.7. Suppose that

$$f(T) = a_n T^n + a_{n+1} T^{n+1} + \cdots \quad \text{and} \quad g(T) = b_m T^m + b_{m+1} T^{m+1} + \cdots$$

are endomorphisms of the group law or bud law F. Say $n \leq m$. Then since F is of the form

$$F(T_1, T_2) = T_1 + T_2 + (\text{higher order terms}),$$

we observe that

$$\begin{aligned}
(f +_F g)(T) = a_n T^n + a_{n+1} T^{n+1} + \cdots + a_{m-1} T^{m-1} \\
+ (a_m + b_m) T^m + (\text{higher order terms}).
\end{aligned}$$

Let $\varphi\colon A \to B$ be any ring homomorphism. If F is a formal group law or bud law over A, then application of φ to the coefficients of F yields a group law or bud law, respectively, $\varphi_* F$ over B. While it is easy to see that *universal* group laws and bud laws exist, Lazard [L] obtained the following remarkable description of the rings over which these universal laws are defined. Let $n \geq 1$.

Theorem 1.1.7 (Lazard). *There exists a universal n-bud law U_n over the polynomial ring $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ and a universal formal group law U over the polynomial ring $\mathbb{Z}[t_1, t_2, \ldots]$.*

In other words, the rings $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ and $\mathbb{Z}[t_1, t_2, \ldots,]$ *corepresent* the functors $A \mapsto \{n\text{-bud laws over } A\}$ and $A \mapsto \{\text{formal group laws over } A\}$, respectively. We will recall a refinement of Lazard's theorem later in (3.1.14).

Proof of (1.1.7). This was first proved, though only implicitly for bud laws, in [L, Théorèmes II and III and their proofs]. One may also consult [Ha, I 5.3.1, 5.7.3, 5.7.4] ([Ha] refers to buds as "chunks"). □

Remark 1.1.8. The choice of U_n and U in the theorem are by no means canonical. That is, $\mathbb{Z}[a_1, \ldots, a_{n-1}]$ and $\mathbb{Z}[a_1, a_2, \ldots,]$ *noncanonically* corepresent the respective functors $A \mapsto \{n\text{-bud laws over } A\}$ and $A \mapsto \{\text{formal group laws over } A\}$. So already one sees an advantage in "dividing out" out by the isomorphisms between the universal laws to obtain more canonical objects, so that no particular choice of universal law is preferred. Hence one is led to the *stacks* of formal Lie groups and of n-buds.

1.2. **Formal Lie varieties.** There are many (equivalent) "geometric" definitions of formal Lie groups in the literature. We shall follow [G, Me] in our treatment. In passing from formal group laws to formal Lie groups, one may take as point of departure the following observation: to give a formal group law over the ring A is to give a group structure on the formal scheme $\mathrm{Spf}\, A[[T]]$ over $\mathrm{Spec}\, A$ with the 0 section as identity; see (1.3.3). In general, a formal Lie group is something modeled locally by this picture; see (1.3.1). Hence the basic geometry of formal Lie groups lies in the realm of *formal* geometry. In this section, we introduce what are, in some sense, the basic formal-geometric objects underlying the formal Lie groups, namely the *formal Lie varieties*.

Let S be a scheme.

Definition 1.2.1. A *(pointed, 1-parameter) formal Lie variety over S* is a sheaf X on $(\mathrm{Sch})_{/S}$ for the fppf topology equipped with a section $\sigma\colon S \to X$, such that, Zariski locally on S, there is an isomorphism of pointed sheaves $X \simeq \mathrm{Spf}\, \mathscr{O}_S[[T]]$, where $\mathrm{Spf}\, \mathscr{O}_S[[T]]$ is pointed by the 0-section. A *morphism* of formal Lie varieties is a morphism of pointed sheaves.

In other words, a (pointed, 1-parameter) formal Lie variety is a pointed formal scheme over S locally (on S) of the form $\mathrm{Spf}\, \mathscr{O}_S[[T]]$.

Remark 1.2.2. More generally, a *pointed formal Lie variety* (without condition on the number of parameters) is a pointed sheaf over S Zariski locally of the form $\mathrm{Spf}\, \mathscr{O}_S[[T_1, \ldots, T_n]]$ for some n locally constant on S [G, III 6.7]. A *formal Lie variety* (without specified point) is a sheaf over S fpqc locally of the form

$$\mathrm{Spf}\, \mathscr{O}_S[[T_1, \ldots, T_n]]$$

for varying n [G, VI 1.3]. Though these generalizations of (1.2.1) are certainly not without their place, we will have little occasion to consider them here. So, for simplicity, we shall abuse language and always understand "formal Lie variety" to mean "pointed, 1-parameter formal Lie variety", unless explicitly stated otherwise.

Example 1.2.3. The most basic and important example of a formal Lie variety over any base S is just $\mathrm{Spf}\,\mathscr{O}_S[[T]]$ itself, equipped with the 0 section, that is, with the map specified on algebras $\mathscr{O}_S[[T]] \to \mathscr{O}_S$ by $T \mapsto 0$. We denote this example by $\widehat{\mathbb{A}}_S$ or, when the base is clear from context, by $\widehat{\mathbb{A}}$. When S is an affine scheme $\mathrm{Spec}\,A$, we also denote $\widehat{\mathbb{A}}_S$ by $\widehat{\mathbb{A}}_A$.

Our notation is nonstandard. It is typical to write $\widehat{\mathbb{A}}^1_S$ for the formal line $\mathrm{Spf}\,\mathscr{O}_S[[T]]$ obtained by completing $\mathbb{A}^1_S$ at the origin. But since our interest is almost exclusively in pointed, 1-parameter formal Lie varieties, we shall suppress the superscript ^{1} to reduce clutter, and we shall always understand $\widehat{\mathbb{A}}_S$ to be equipped with the zero section.

Example 1.2.4. More generally, if T is any smooth scheme of relative dimension 1 over S and $S \to T$ is a section, then the completion of T along the section is a formal Lie variety over S.

Remark 1.2.5. One can give a more intrinsic version of the definition of formal Lie variety: see [G, VI 1.3] or [Me, II 1.1.4]. But (1.2.1) has the advantage of being reasonably concrete, and it will certainly suffice for our purposes.

1.3. **Formal Lie groups.** Let S be a scheme.

Definition 1.3.1. A *(commutative, 1-parameter) formal Lie group over S* is an fppf sheaf of commutative groups on $(\mathrm{Sch})_{/S}$ such that the underlying pointed sheaf of sets is a formal Lie variety (1.2.1).

In other words, a formal Lie group is a formal Lie variety (X, σ) made into a commutative group object in the category of sheaves on $(\mathrm{Sch})_{/S}$, such that the given section σ is the identity section.

Remark 1.3.2. One can certainly formulate (1.3.1) without the commutativity condition. And, analogously to (1.2.2), one may define a *formal Lie group* in full generality as an fppf sheaf of (not-necessarily-commutative) groups over S whose underlying pointed sheaf of sets is a formal Lie variety in the general sense of (1.2.2). But as before, we shall have no use for the more general notion. So we shall abuse language and always use "formal Lie group" in the sense stated in (1.3.1).

Example 1.3.3. Consider the formal Lie variety $\widehat{\mathbb{A}}$ over S (1.2.3). To make $\widehat{\mathbb{A}}$ into a formal Lie group, we must define a multiplication map $\widehat{\mathbb{A}} \times_S \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$. Since

$$\widehat{\mathbb{A}} \times_S \widehat{\mathbb{A}} \cong \mathrm{Spf}\,\mathscr{O}_S[[T_1, T_2]],$$

we may equivalently define a continuous map of $\mathscr{O}_S$-algebras

$$\mathscr{O}_S[[T_1, T_2]] \longleftarrow \mathscr{O}_S[[T]].$$

Any such map is determined by the image $F(T_1, T_2)$ of T in the global sections $\Gamma(S)[[T_1, T_2]]$. Then $\widehat{\mathbb{A}}$ becomes a formal Lie group with the 0 section as identity exactly when F is a formal group law over $\Gamma(S)$ in the classical sense (1.1.1). Hence, to give a formal group law is to give a formal Lie group with a choice of coordinate. We write $\widehat{\mathbb{A}}^F = \widehat{\mathbb{A}}^F_S$ for the group structure on $\widehat{\mathbb{A}}$ obtained from F.

Example 1.3.4. Let F and G be formal group laws over $\Gamma(S)$. Then a morphism of formal Lie varieties $f\colon \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$ is a morphism of formal Lie groups $f\colon \widehat{\mathbb{A}}^F \to \widehat{\mathbb{A}}^G$ exactly when the diagram of maps on global sections

$$
\begin{array}{ccc}
\Gamma(S)[[T_1,T_2]] & \xleftarrow{\ (f\times f)^{\#}\ } & \Gamma(S)[[T_1,T_2]] \\
{\scriptstyle F(T_1,T_2)}\big\uparrow & & \big\uparrow{\scriptstyle G(T_1,T_2)} \\
\Gamma(S)[[T]] & \xleftarrow{\ \ f^{\#}\ \ } & \Gamma(S)[[T]]
\end{array}
$$

commutes, that is, when

$$f^{\#}\big(F(T_1,T_2)\big) = G\big(f^{\#}(T_1), f^{\#}(T_2)\big).$$

Hence we recover the classical notion of a group law homomorphism $F \to G$ (1.1.1).

Example 1.3.5.

- The *additive formal Lie group* $\widehat{\mathbb{G}}_a = \widehat{\mathbb{G}}_{a,S}$ *over* S is $\widehat{\mathbb{A}}_S^F$ for
$$F(T_1,T_2) = T_1 + T_2$$
  the additive group law (1.1.3). $\widehat{\mathbb{G}}_a$ is the completion of $\mathbb{G}_a$ at the identity.
- The *multiplicative formal Lie group* $\widehat{\mathbb{G}}_m = \widehat{\mathbb{G}}_{m,S}$ *over* S is $\widehat{\mathbb{A}}_S^F$ for
$$F(T_1,T_2) = T_1 + T_2 + T_1 T_2$$
  the multiplicative group law (1.1.3). $\widehat{\mathbb{G}}_m$ is the completion of $\mathbb{G}_m$ at the identity.
- If E is an elliptic curve over S, then the completion of E at the identity is a formal Lie group over S. Note that this furnishes many examples of formal Lie groups not admitting a global coordinate.
- More generally, completion at the identity of any smooth commutative group scheme of relative dimension 1 yields a formal Lie group. When S is Spec of an algebraically closed field, then $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves are the *only* such connected group schemes.

At this juncture, we could perfectly well begin to consider the moduli stack of formal Lie groups. But, as noted in the introduction, it turns out that this stack is not algebraic. So in the next section, we shall begin laying the groundwork to study the related moduli problem of classifying *n-buds*; this will afford an "algebraic approximation" to the moduli stack of formal Lie groups, in a sense we make precise in (2.6.8). We shall return to the moduli stacks of formal Lie varieties and of formal Lie groups in §2.5 and §2.6, respectively.

1.4. **Infinitesimal neighborhoods.** Roughly speaking, in algebra, a bud law is a truncation of a formal group law. In the geometric setting, the role of truncation will be played by *taking infinitesimal neighborhoods*. In other words, roughly speaking, a bud will be an infinitesimal neighborhood of the identity of a formal Lie group. Hence the basic geometry of buds lies in the realm of *infinitesimal* geometry. In this section we introduce some of the basic language surrounding these ideas.

Let us begin by recalling the notion of infinitesimal neighborhood from [G, VI 1.1] or [Me, II 1.01]. Let S be a base scheme, Y and X sheaves on $(\mathrm{Sch})_{/S}$ for some fixed topology between the fpqc and Zariski topologies, inclusive, and $Y \hookrightarrow X$ a monomorphism. Let $n \geq 0$.

Definition 1.4.1. The *nth infinitesimal neighborhood of Y in X* is the subsheaf of X obtained as the sheafification of the subpresheaf of X defined on points by

$$T \longmapsto \left\{ T \to X \;\middle|\; \begin{array}{c} \text{there exists a commutative diagram} \\[4pt] \begin{array}{ccc} T' & \hookrightarrow & T \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array} \\[4pt] \text{for some closed subscheme } T' \text{ of } T \text{ defined by a} \\ \text{quasi-coherent ideal } \mathscr{I} \subset \mathscr{O}_T \text{ satisfying } \mathscr{I}^{n+1} = 0. \end{array} \right\}.$$

We denote the nth infinitesimal neighborhood by $\mathrm{Inf}_Y^n(X)$ or, when Y is clear from context, by $X^{(n)}$.

Remark 1.4.2. When $Y \hookrightarrow X$ is a closed immersion of schemes, say with associated sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$, one typically defines $\mathrm{Inf}_Y^n(X)$ to be the closed subscheme of X whose underlying topological space is the image of Y and whose structure sheaf is the restriction of $\mathscr{O}_X/\mathscr{I}^{n+1}$. It is not hard to verify that, in this situation, this notion of Inf and the notion of (1.4.1) agree; see [Me, II 1.02].

Remark 1.4.3. One verifies at once that Inf is *functorial* in X and Y: precisely, any commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \hookrightarrow & X', \end{array}$$

in which the horizontal arrows are monomorphisms, induces a map $\mathrm{Inf}_Y^n(X) \to \mathrm{Inf}_{Y'}^n(X')$ for any n.

Remark 1.4.4. Formation of $\mathrm{Inf}_Y^n(X)$ is compatible with base change on S by [Me, II 1.03].

We will be especially interested in infinitesimal neighborhoods in the case $Y = S$, so that X is *pointed*. In this situation, we have the following definition.

Definition 1.4.5. X is *n-infinitesimal* (resp., *ind-infinitesimal*) if the natural arrow $X^{(n)} \to X$ (resp., $\varinjlim_n X^{(n)} \to X$) is an isomorphism. We denote by $(n\text{-Inf})(S)$ (resp., $(\infty\text{-Inf})(S)$) the category of n-infinitesimal (resp., ind-infinitesimal) sheaves over S.

Example 1.4.6. The motivation for introducing (1.4.5) is simply that any formal Lie variety is ind-infinitesimal, and any nth infinitesimal neighborhood of a pointed sheaf is n-infinitesimal.

Remark 1.4.7. Of course, one has the obvious implications

$$n\text{-infinitesimal} \implies m\text{-infinitesimal for } m \geq n \implies \text{ind-infinitesimal}.$$

Remark 1.4.8. One verifies at once that the properties of being n-infinitesimal or ind-infinitesimal are stable under base change. Hence $(n\text{-Inf})$ and $(\infty\text{-Inf})$ define fibered categories over (Sch).

Remark 1.4.9. As a special case of (1.4.3), the assignment $X \mapsto X^{(n)}$ specifies a functor

$$\left\{ \begin{array}{c} \text{pointed fppf} \\ \text{sheaves on } (\mathrm{Sch})_{/S} \end{array} \right\} \longrightarrow (n\text{-Inf})(S).$$

Hence we get a natural "truncation" functor

$$(\infty\text{-Inf})(S) \longrightarrow (n\text{-Inf})(S).$$

We shall return to the topic of truncation a bit more in earnest in §2.4.

Remark 1.4.10. If X and Y are n- and m-infinitesimal, respectively, sheaves over S, then the product (in the category of pointed sheaves over S) $X \times_S Y$ is $(n+m)$-infinitesimal. Hence a product of ind-infinitesimal sheaves is ind-infinitesimal.

On the other hand, $(n\text{-Inf})(S)$ itself contains finite products: namely, the product of sheaves X and Y in $(n\text{-Inf})(S)$ is $(X \times_S Y)^{(n)}$.

1.5. **Germs.** In this section we introduce the "truncated" analogs of the formal Lie varieties, namely the *germs*; these will hold the same relation to buds that the formal Lie varieties hold to formal Lie groups. To define n-germs, we could, so to speak, just "take the nth infinitesimal neighborhood" of the definition of formal Lie variety (1.2.1); that is, we could copy (1.2.1) word-for-word, only replacing $\mathscr{O}_S[[T]]$ everywhere with $\mathscr{O}_S[T]/(T)^{n+1}$. Instead, we shall give a somewhat more intrinsic definition. We shall deduce that our definition amounts to the infinitesimal version of a formal Lie variety in (1.6.4).

Let $n \geq 0$, and let S be a scheme.

Definition 1.5.1. A *(pointed, 1-parameter) n-germ over S* is a pointed, representable, n-infinitesimal (1.4.5) sheaf X over S satisfying

(LF1) if $\mathscr{I}$ is the sheaf of ideals on X associated to the given section $\sigma\colon S \to X$, then each successive quotient $\sigma^*(\mathscr{I}^i/\mathscr{I}^{i+1})$ for $i = 1, \ldots, n$ is a locally free $\mathscr{O}_S$-module of rank 1.

We often denote n-germs as triples (X, π, σ), where π denotes the structure map $X \to S$ and σ denotes the section.

It is convenient to allow the $n = 0$ case; then (LF1) is vacuous, and a 0-germ over S consists just of an isomorphism $X \xrightarrow{\sim} S$.

Remark 1.5.2. One could give a fully general definition of n-germs in a way entirely analogous to (1.2.2). But, just as in (1.2.2), we won't have occasion to consider the more general notion. So we shall abuse language and always use "n-germ" in the sense stated in (1.5.1).

Example 1.5.3. The most basic and important example of an n-germ over S is just the infinitesimal version of $\widehat{\mathbb{A}}_S$ (1.2.3),

$$\mathbb{T} := \mathbb{T}_S := \mathbb{T}_{n,S} := \mathrm{Spec}\, \mathscr{O}_S[T]/(T)^{n+1},$$

equipped with the 0 section. In our discussion of germs, we will often work with respect to a fixed nonnegative integer n and/or a fixed base S, and, accordingly, we will often favor the plainer notation $\mathbb{T}$ or $\mathbb{T}_S$ over $\mathbb{T}_{n,S}$, provided no confusion seems likely. When S is an affine scheme $\mathrm{Spec}\, A$, we also denote $\mathbb{T}_S$ by $\mathbb{T}_A$.

Our choice of the letter $\mathbb{T}$ is for "trivial"; see (1.6.1).

Remark 1.5.4. Note that for $m \geq n$, we have $\widehat{\mathbb{A}}_S^{(n)} \cong \mathbb{T}_{m,S}^{(n)} \cong \mathbb{T}_{n,S}$. More generally, any truncation of a formal Lie variety or of a germ is a germ.

Remark 1.5.5. Taking $i = 1$ in the definition (1.5.1), we see that, in particular, the conormal sheaf associated to σ is a line bundle on S. This line bundle will play an important role in our later study of the *height stratification*. We remark in addition that, since σ is a section of π, the conormal sheaf is canonically isomorphic to $\sigma^*(\Omega_{X/S}^1)$ [EGAIV$_4$, 17.2.5].

Remark 1.5.6. It is immediate that the morphisms π and σ in (1.5.1) are inverse homeomorphisms of topological spaces. Hence each sheaf $\sigma^*(\mathscr{I}^i/\mathscr{I}^{i+1})$ appearing in condition (LF1) is canonically isomorphic to $\pi_*(\mathscr{I}^i/\mathscr{I}^{i+1})$.

Remark 1.5.7. In the language of [G, Me], we could have defined an n-germ over S as an object of $(n\text{-Inf})(S)$, representable and smooth to order n along the section [G, VI 1.2; Me, II 3.1.2], such that the conormal sheaf is a line bundle on S. The equivalence of this notion with that of our definition (1.5.1), if not immediately apparent, will be made clear in (1.6.4).

Definition 1.5.8. We define $(n\text{-germs})(S)$ to be the full subcategory of pointed sheaves on $(\mathrm{Sch})_{/S}$ consisting of the n-germs. We define $\mathscr{G}_n(S)$ to be the groupoid of n-germs and their isomorphisms.

It follows from (1.5.10) below that $(n\text{-germs})$ and $\mathscr{G}_n$ define a fibered category and a CFG, respectively, over (Sch). For the moment, consider the general situation of a scheme X over S and a quasi-coherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$.

Lemma 1.5.9 (compare [EGAIV$_4$, 16.2.4]). *Let $i \geq 1$, and suppose $\mathscr{I}^j/\mathscr{I}^{j+1}$ is S-flat for $0 \leq j \leq i - 1$. Then formation of $\mathscr{I}^i$ is compatible with base change on S; that is, for any Cartesian square*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ p\ } & X \\
\downarrow & & \downarrow \\
S' & \longrightarrow & S,
\end{array}
$$

$\mathscr{I}' := p^*\mathscr{I}$ *is naturally identified with a sheaf of ideals in $\mathscr{O}_{X'}$, and the natural arrow $p^*(\mathscr{I}^i) \to (\mathscr{I}')^i$ is an isomorphism.*

Proof. Consider the exact sequence

$$
\mathscr{I}^{\otimes i} \xrightarrow{\ m\ } \mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{I}^i \longrightarrow 0,
$$

where m is the multiplication map. Now, $\mathscr{O}_X/\mathscr{I}^i$ admits a filtration

$$
\mathscr{O}_X/\mathscr{I}^i \supset \mathscr{I}/\mathscr{I}^i \supset \mathscr{I}^2/\mathscr{I}^i \supset \cdots \supset \mathscr{I}^{i-1}/\mathscr{I}^i \supset 0
$$

whose successive quotients are S-flat by hypothesis. Hence $\mathscr{O}_X/\mathscr{I}^i$ is S-flat. That is, m has flat cokernel. Hence formation of the image of m is compatible with base change. The lemma follows. $\qquad\square$

Lemma 1.5.10. *Suppose X is an n-germ over S. Then for any base change $S' \to S$, $S' \times_S X$ is an n-germ over S'.*

Proof. Immediate from the previous lemma, since base change for arbitrary schemes preserves quotients of quasi-coherent modules and the property of being finite locally free of given rank. $\qquad\square$

1.6. **Trivial germs.** Let us begin with the definition.

Definition 1.6.1. The n-germ X over S is *trivial* if it is isomorphic to $\mathbb{T}_S$ (1.5.3).

We shall now show that every germ is trivial Zariski locally. As a first step, let $\pi\colon X \to S$ be a scheme over S admitting a section σ which is a closed immersion and a homeomorphism.

Lemma 1.6.2. X *is affine over* S.

Proof. Apply [SGA3$_\mathrm{I}$, VI$_\mathrm{B}$ 2.9.1], which says that for any morphism of schemes π, if π is a homeomorphism between the underlying topological spaces, then π is an affine morphism. Alternatively, we can give a simple direct proof in the present situation by using that the inverse homeomorphism σ is a closed immersion. Consider an open affine V in X. Since a closed immersion is an affine morphism, $U := \sigma^{-1}(V)$ is affine in S. Then $\pi^{-1}(U) = V$ is affine. But we can certainly cover S with open affines of the form U, so we're done. $\square$

Let us now specialize to the case that (X, π, σ) is an n-germ.

Lemma 1.6.3. *Let* $\mathscr{I} \subset \mathscr{O}_X$ *be the sheaf of ideals corresponding to* σ. *Then the natural map*

$$(\mathscr{I}/\mathscr{I}^2)^{\otimes i} \longrightarrow \mathscr{I}^i/\mathscr{I}^{i+1}$$

is an isomorphism for $1 \leq i \leq n$.

Proof. The map is an epimorphism of locally free sheaves of the same rank, hence an isomorphism. $\square$

Proposition 1.6.4. *Zariski locally on* S, X *is trivial.*

Proof. Localizing on S, we may assume by (1.6.2) that $S = \operatorname{Spec} A$, $X = \operatorname{Spec} B$, the composition

$$A \xleftarrow{\sigma^\#} B \xleftarrow{\pi^\#} A$$

is id_A, and $I := \ker[B \to A]$ is an ideal such that $I^{n+1} = 0$ and I/I^2 is a free A-module of rank 1.

Let $\bar{t}$ generate I/I^2 as an A-module and choose any lift $t \in I$. We claim that the map $\varphi\colon A[T]/(T)^{n+1} \to B$ sending $T \mapsto t$ is an isomorphism. Indeed, φ is well-defined because $t \in I$ and $I^{n+1} = 0$. Let us consider $A[T]/(T)^{n+1}$ as filtered T-adically, and B as filtered I-adically. Then φ is a map of filtered rings. But by (1.6.3), φ induces an isomorphism on associated graded algebras. Hence φ is an isomorphism. $\square$

In light of the proposition, it will be important later to understand the automorphisms of $\mathbb{T}_S$.

Definition 1.6.5. We define $\mathscr{E}nd(\mathbb{T}_n)$ to be the presheaf of monoids on (Sch)

$$\mathscr{E}nd(\mathbb{T}_n)\colon S \longmapsto \operatorname{End}_{(n\text{-germs})(S)}(\mathbb{T}_{n,S})$$

and $\mathscr{A}ut(\mathbb{T}_n)$ to be the presheaf of groups on (Sch)

$$\mathscr{A}ut(\mathbb{T}_n)\colon S \longmapsto \operatorname{Aut}_{(n\text{-germs})(S)}(\mathbb{T}_{n,S}).$$

Fix a base scheme S. To give an endomorphism of $\mathbb{T}_S$ is to give a map of augmented $\mathscr{O}_S$-algebras $\mathscr{O}_S[T]/(T)^{n+1} \to \mathscr{O}_S[T]/(T)^{n+1}$; this, in turn, is given by the image $a_1 T + \cdots + a_n T^n$ of T in $\Gamma_n(S;T)$. The endomorphism of $\mathbb{T}_S$ is then an automorphism exactly when $a_1 T + \cdots + a_n T^n$ is invertible under composition in $\Gamma_n(S;T)$, that is, when $a_1 \in \Gamma(S)^{\times}$. We have shown the following.

Proposition 1.6.6. *As set-valued functors, $\mathscr{E}nd(\mathbb{T}_n)$ is canonically represented by*

$$\operatorname{Spec} \mathbb{Z}[a_1, a_2, \ldots, a_n] = \mathbb{A}^n_{\mathbb{Z}},$$

and $\mathscr{A}ut(\mathbb{T}_n)$ is canonically represented by the open subscheme

$$\operatorname{Spec} \mathbb{Z}[a_1, a_1^{-1}, a_2, \ldots, a_n]$$

of $\mathbb{A}^n_{\mathbb{Z}}$. $\hfill\square$

Explicitly, the monoid structure on $\operatorname{Spec} \mathbb{Z}[a_1, \ldots, a_n] = \mathbb{A}^n_{\mathbb{Z}}$ obtained from the identification with $\mathscr{E}nd(\mathbb{T}_n)$ is given by composition of polynomials $a_1 T + \cdots + a_n T^n$ in the truncated polynomial ring $\mathbb{Z}[a_1, \ldots, a_n][T]/(T)^{n+1}$.

Remark 1.6.7. $\mathscr{A}ut(\mathbb{T}_n)$ admits a decreasing filtration of closed sub-group schemes

$$\mathscr{A}ut(\mathbb{T}_n) =: \mathscr{A}_0^{\mathbb{T}_n} \supset \mathscr{A}_1^{\mathbb{T}_n} \supset \cdots \supset \mathscr{A}_{n-1}^{\mathbb{T}_n} \supset \mathscr{A}_n^{\mathbb{T}_n} := 1$$

defined on S-valued points by

$$\mathscr{A}_i^{\mathbb{T}_n}(S) = \{\, T + a_{i+1} T^{i+1} + \cdots + a_n T^n \mid a_{i+1}, \ldots, a_n \in \Gamma(S) \,\}, \quad 1 \le i \le n-1.$$

Said differently, $\mathscr{A}_i^{\mathbb{T}_n}$ is just the kernel of the homomorphism $\mathscr{A}ut(\mathbb{T}_n) \to \mathscr{A}ut(\mathbb{T}_i)$ induced by the identification $\mathbb{T}_n^{(i)} \cong \mathbb{T}_i$, $0 \le i \le n$. One verifies at once that the map on points

$$T + a_{i+1} T^{i+1} + \cdots + a_n T^n \longmapsto a_{i+1}$$

specifies an isomorphism of $\mathbb{Z}$-groups

$$\mathscr{A}_i^{\mathbb{T}_n}/\mathscr{A}_{i+1}^{\mathbb{T}_n} \overset{\sim}{\longrightarrow} \begin{cases} \mathbb{G}_m, & i = 0; \\ \mathbb{G}_a, & 1 \le i \le n-1. \end{cases}$$

We shall return to the $\mathscr{A}_i^{\mathbb{T}_n}$'s in §3.7.

1.7. **Buds.** We now introduce the infinitesimal versions of the formal Lie groups, namely the *buds*; these are the algebro-geometric analogs of the bud laws. Let S be a scheme and $n \ge 0$.

Definition 1.7.1. A *(commutative, 1-parameter) n-bud over S* consists of an n-germ X over S (1.5.1) equipped with a morphism of S-schemes, which we think of as a multiplication map,

$$F \colon (X \times_S X)^{(n)} \longrightarrow X,$$

satisfying the constraints

    (I) (identity) the section $S \xrightarrow{\sigma} X$ is a left and right identity for F, i.e. both compositions in the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{(\sigma\pi, \mathrm{id}_X)} & (X \times_S X)^{(n)} \\
{\scriptstyle (\mathrm{id}_X, \sigma\pi)} \downarrow & & \downarrow {\scriptstyle F} \\
(X \times_S X)^{(n)} & \xrightarrow{\quad F \quad} & X
\end{array}
$$

        equal id_X;

(A) (associativity) F is associative on points of $(X \times_S X \times_S X)^{(n)}$, i.e. the restrictions of $F \times \mathrm{id}_X$ and $\mathrm{id}_X \times F$ to $(X \times_S X \times_S X)^{(n)}$ yield a commutative diagram

$$
\begin{array}{ccc}
(X \times_S X \times_S X)^{(n)} & \xrightarrow{\ \mathrm{id}_X \times F\ } & (X \times_S X)^{(n)} \\
{\scriptstyle F \times \mathrm{id}_X}\big\downarrow & & \big\downarrow{\scriptstyle F} \\
(X \times_S X)^{(n)} & \xrightarrow{\qquad F \qquad} & X;
\end{array}
$$

and

(C) (commutativity) F is commutative, i.e. letting $\tau \colon X \times_S X \to X \times_S X$ denote the transposition map $(x, y) \mapsto (y, x)$ and restricting τ to $(X \times_S X)^{(n)}$, F is τ-equivariant, i.e. the diagram

$$
\begin{array}{ccc}
(X \times_S X)^{(n)} & \xrightarrow{\ \ \tau\ \ } & (X \times_S X)^{(n)} \\
 & {\scriptstyle F}\searrow \quad \swarrow {\scriptstyle F} & \\
 & X &
\end{array}
$$

commutes.

Of course, the infinitesimal neighborhoods in the definition are all taken with respect to the sections induced by σ. It is an immediate consequence of the functoriality of Inf that the diagrams in the definition are well defined.

Remark 1.7.2. Heuristically, the multiplication map F in the definition is almost a commutative monoid scheme structure on X over S, but F is defined only on points of a certain subfunctor of the product $X \times_S X$. On the other hand, recall from (1.4.10) that $(X \times_S X)^{(n)}$ is the honest product of X with itself in the category $(n\text{-Inf})(S)$. Hence n-buds over S are commutative monoids in $(n\text{-Inf})(S)$. In fact, we'll verify in §1.8 that the n-buds are precisely the n-germs endowed with a commutative *group* structure in $(n\text{-Inf})(S)$.

Remark 1.7.3. Just as we noted for formal Lie groups (1.3.2), one can certainly consider n-buds on many parameters, or without the commutativity constraint (C). But as always, we shall only be concerned with the commutative, 1-parameter case. So we shall always abusively understand "n-bud" in the sense of (1.7.1), except where explicitly stated otherwise.

Remark 1.7.4. In the trivial case $n = 0$, we have $X \xrightarrow{\sim} S$ and F is forced to be the canonical isomorphism $(X \times_S X)^{(0)} \xrightarrow{\sim} X$.

Example 1.7.5. Consider the n-germ $\mathbb{T}$ over S (1.5.3). Then, quite analogously to (1.3.3), to give an n-bud structure $(\mathbb{T} \times_S \mathbb{T})^{(n)} \to \mathbb{T}$ with the given section as identity is to give an n-bud law F over $\Gamma(S)$ in the classical sense (1.1.2). Hence, to give an n-bud law is to give an n-bud with a choice of coordinate. We write $\mathbb{T}^F = \mathbb{T}_S^F$ for the bud structure on $\mathbb{T}$ obtained from F.

Example 1.7.6. Everything in (1.3.5) admits an obvious analog for buds. In particular, let us signal the following.

- The *additive n-bud* $\mathbb{G}_a^{(n)} = \mathbb{G}_{a,S}^{(n)}$ *over* S is the nth infinitesimal neighborhood of $\mathbb{G}_a$ at the identity, that is, the n-bud $\mathbb{T}_{n,S}^F$ for

$$
F(T_1, T_2) = T_1 + T_2
$$

the additive bud law (1.1.3).

- The *multiplicative n-bud* $\mathbb{G}_m^{(n)} = \mathbb{G}_{m,S}^{(n)}$ *over S* is the nth infinitesimal neighborhood of $\mathbb{G}_m$ at the identity, that is, the n-bud $\mathbb{T}_S^F$ for

$$F(T_1, T_2) = T_1 + T_2 + T_1 T_2$$

the multiplicative bud law (1.1.3).

Remark 1.7.7. For F a group law (resp., m-bud law with $m \geq n$), let us denote by $F^{(n)}$ the n-bud law obtained from F by discarding terms of degree $\geq n+1$. Then we have $(\widehat{\mathbb{A}}^F)^{(n)} \cong \mathbb{T}_n^{F^{(n)}}$ (resp., $(\mathbb{T}_m^F)^{(n)} \cong \mathbb{T}_n^{F^{(n)}}$). More generally, any truncation of a formal Lie group or of a bud is a bud, since truncation $(\infty\text{-Inf})(S) \to (n\text{-Inf})(S)$ and $(m\text{-Inf})(S) \to (n\text{-Inf})(S)$, $m \geq n$, preserves finite (including empty) products.

There is an obvious notion of morphism of buds.

Definition 1.7.8. A *morphism* $f \colon X \to Y$ of n-buds over S is a morphism of monoid objects in $(n\text{-Inf})(S)$, that is, a morphism of the underlying n-germs such that

$$
\begin{array}{ccc}
(X \times_S X)^{(n)} & \xrightarrow{(f \times f)^{(n)}} & (Y \times_S Y)^{(n)} \\
\downarrow & & \downarrow \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

commutes.

Example 1.7.9. Analogously to (1.3.4), to give a morphism of buds $\mathbb{T}_S^F \to \mathbb{T}_S^G$ over S is to give a homomorphism of bud laws $F \to G$ (1.1.2).

Definition 1.7.10. We define $(n\text{-buds})(S)$ to be the category of n-buds and bud morphisms over S. We define $\mathscr{B}_n(S)$ to be the groupoid of n-buds and bud isomorphisms over S.

Remark 1.7.11. Given an n-bud X over S and a base change $f \colon S' \to S$, we know $X' := S' \times_S X$ is at least an n-germ by (1.5.10). But then the base change of the multiplication map makes X' into a bud, since infinitesimal neighborhoods (1.4.4) and fibered products are compatible with base change. Hence $(n\text{-buds})$ and $\mathscr{B}_n$ define a fibered category and a CFG, respectively, over (Sch).

Note that when $X = \mathbb{T}_S^F$, one has $X' \cong \mathbb{T}_{S'}^{F'}$, where F' is the bud law over $\Gamma(S')$ obtained by applying $f^\#$ to the coefficients of F.

1.8. **Buds as group objects.** Fix a base scheme S. We remarked in (1.7.2) that n-buds over S are almost commutative monoids in the category of S-schemes, and are honest commutative monoids in the category $(n\text{-Inf})(S)$ (1.4.5). As promised, we'll now see that n-buds are honest *group* objects in $(n\text{-Inf})(S)$.

Proposition 1.8.1. *The n-buds over S are precisely the n-germs over S endowed with a commutative group structure in $(n\text{-Inf})(S)$. The n-bud morphisms over S are precisely the homomorphisms of group objects in $(n\text{-Inf})(S)$.*

Proof. All we need to show is that every n-bud X is automatically equipped with an inverse morphism $X \to X$. Since the inverse is unique if it exists, it suffices to find the inverse locally on S. Hence we may assume X has trivial underlying germ (1.6.4), so that we may assume $X = \mathbb{T}_S^F$ for some n-bud law $F(T_1, T_2) \in \Gamma_n(S; T_1, T_2)$ (1.7.5). Now use (1.1.4, 1.7.9). $\qquad\square$

We obtain the following as a formal consequence.

Corollary 1.8.2. *For n-buds X and Y over S, the set of bud morphisms $X \to Y$ is naturally an abelian group. Moreover, composition of bud morphisms is bilinear.*
□

Explicitly, bud morphisms $X \to Y$ are added as elements of $\mathrm{Hom}_{(n\text{-Inf})(S)}(X, Y)$ under the group structure coming from Y. The content of the corollary is that bud morphisms form a *subgroup* of $\mathrm{Hom}_{(n\text{-Inf})(S)}(X, Y)$.

Remark 1.8.3. The category of n-buds over S is not additive for $n \geq 1$, since the product of n-germs, whether taken in the category of pointed sheaves or $(n\text{-Inf})(S)$, is not again an n-germ. But the problem is only that we've restricted to the 1-parameter case: commutative n-buds in the general sense of (1.7.3), with no constraint on the number of parameters, do form an additive category.

2. Basic moduli theory

We now begin to consider the basic moduli theory of the stacks of n-germs, n-buds, formal Lie varieties, and formal Lie groups.

2.1. The stack of n-germs. In this section we show that the moduli stack of n-germs $\mathscr{G}_n$, $n \geq 1$ is equivalent to the classifying algebraic stack $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$, with $\mathscr{A}ut(\mathbb{T}_n)$ as defined in (1.6.5).

Proposition 2.1.1. *$\mathscr{G}_n$ is a stack over* (Sch) *for the fpqc topology.*

Proof. We have to check that objects and morphisms descend. It is clear that $\mathscr{G}_n$ is a stack for the Zariski topology since we can glue schemes and morphisms of schemes. So we may restrict to the case of a base scheme S and a faithfully flat quasi-compact morphism $f \colon S' \to S$. Our argument from here will be a straightforward application of the descent theory in [SGA1, VIII].

Descent for morphisms of germs along f is an immediate consequence of descent for morphisms of schemes [SGA1, VIII 5.2]. To check descent for objects, let X' be an n-germ over S' equipped with a descent datum. Then X' is affine over S' (1.6.2). So X' descends to a scheme X affine over S [SGA1, VIII 2.1]. By descent for morphisms, the section for X' descends to a section σ for X. It remains to show that X is n-infinitesimal along σ and that constraint (LF1) in the definition of n-germ (1.5.1) holds for X. We may assume $X = \mathrm{Spec}\,\mathscr{A}$ for some quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$. Let $\mathscr{I} := \ker[\mathscr{A} \to \mathscr{O}_S]$ be the ideal corresponding to σ. Then $\mathscr{A} \cong \mathscr{O}_S \oplus \mathscr{I}$, so $\mathscr{I}$ certainly pulls back to the ideal corresponding to the given section $S' \to X'$. Moreover, since arbitrary base change preserves quotients of quasi-coherent modules, and flat base change preserves powers of ideals, we conclude that the pullback of $\mathscr{I}^i/\mathscr{I}^{i+1}$, $i = 1, \ldots, n$, is locally free of rank 1, and that the pullback of $\mathscr{I}^{n+1}$ is 0. Since $S' \to S$ is faithfully flat, we conclude that $\mathscr{I}^i/\mathscr{I}^{i+1}$ itself is locally free of rank 1 [SGA1, VIII 1.10], and that $\mathscr{I}^{n+1} = 0$. □

Theorem 2.1.2. *$\mathscr{G}_n$ is equivalent to the classifying stack $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$. In particular, $\mathscr{G}_n$ is algebraic.*

Proof. By (1.6.4) and the previous proposition, $\mathscr{G}_n$ is a gerbe (for any topology between the Zariski and fpqc topologies, inclusive) over $\mathrm{Spec}\,\mathbb{Z}$. Moreover, $\mathbb{T}_{\mathbb{Z}}$ (1.5.3) is an object of $\mathscr{G}_n$ over $\mathrm{Spec}\,\mathbb{Z}$, so that $\mathscr{G}_n$ is neutral. The first assertion now follows from [LMB, 3.21].

As for algebraicity, we need note only that $\mathscr{A}ut(\mathbb{T}_n)$ is a smooth, separated group scheme of finite presentation over $\mathbb{Z}$ by (1.6.6), and that the quotient of any algebraic space by such a group scheme is algebraic [LMB, 4.6.1]. (More generally, one may weaken the smoothness hypothesis by requiring only that, in its place, G be flat and of finite presentation [LMB, 10.13.1].) $\qquad\square$

Remark 2.1.3. Of course, for an arbitrary group sheaf G on a site $\mathscr{C}$, the stack $B(G)$ depends on the topology on $\mathscr{C}$. By the theorem, $B\big(\mathscr{A}ut(\mathbb{T}_n)\big)$ is independent of the choice of topology on (Sch) between the Zariski and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(\mathbb{T}_n)$ is in fact a Zariski-torsor.

2.2. **Bud structures on trivial germs.** To prepare for our discussion of the moduli stack of n-buds in the next section, recall from (1.6.4) that every bud has locally trivial underlying germ. Hence the classification of bud structures on $\mathbb{T}_S$ (1.5.3) assumes an important role in the theory. Let $n \geq 1$.

Definition 2.2.1. We define L_n to be the presheaf of sets on (Sch)

$$L_n \colon S \longmapsto \{n\text{-bud structures on } \mathbb{T}_{n,S}\}.$$

Fix a base scheme S. By (1.7.5), to give an n-bud structure on $\mathbb{T}_S$ is to give an n-bud law $F \in \Gamma_n(S; T_1, T_2)$. So by Lazard's theorem (1.1.7), $L_n \simeq \mathbb{A}_{\mathbb{Z}}^{n-1}$ for $n \geq 1$, but the isomorphism is not canonical. In the trivial case $n = 0$, we have $L_0 \cong \operatorname{Spec} \mathbb{Z}$.

We have chosen the notation L_n in honor of Lazard.

Remark 2.2.2. The functor $\mathscr{A}ut(\mathbb{T}_n)$ (1.6.5) acts naturally on L_n as "changes of coordinate": given a bud structure $\mathbb{T}_S^F$ and a germ automorphism f of $\mathbb{T}_S$, transport of structure along f determines a bud structure $\mathbb{T}_S^G$, and f is tautologically a bud isomorphism $\mathbb{T}_S^F \xrightarrow{\sim} \mathbb{T}_S^G$. Explicitly, denoting by $f^{\#}$ the map on global sections of $\mathbb{T}_S$, we have $G(T_1, T_2) = f^{\#}\big[F\big(f^{\#-1}(T_1), f^{\#-1}(T_2)\big)\big]$.

2.3. **The stack of n-buds.** In this section we show that the moduli stack of n-buds $\mathscr{B}_n$, $n \geq 1$, is equivalent to the quotient algebraic stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$, with the schemes $\mathscr{A}ut(\mathbb{T}_n)$ and L_n as defined in (1.6.5) and (2.2.1), respectively, and with the action of $\mathscr{A}ut(\mathbb{T}_n)$ on L_n as described in (2.2.2).

Proposition 2.3.1. $\mathscr{B}_n$ is a stack over (Sch) for the fpqc topology.

Proof. As in (2.1.1), we have to check that morphisms and objects descend. The only new ingredient for n-buds is the multiplication map.

Descent for morphisms of buds is an immediate consequence of descent for morphisms of schemes [SGA1, VIII 5.2] and of compatibility of infinitesimal neighborhoods (1.4.4) and of fibered products with base change, which imply that a germ morphism locally compatible with the multiplication maps is globally compatible.

For objects, suppose we're given an n-bud fpqc-locally on S equipped with a descent datum. By (2.1.1), we get an n-germ X over S. Again using that infinitesimal neighborhoods and fibered products are compatible with base change, the local multiplication maps descend to a multiplication map on X, and we're done. $\qquad\square$

Theorem 2.3.2. $\mathscr{B}_n$ is equivalent to the quotient stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$. In particular, $\mathscr{B}_n$ is algebraic.

Proof. We shall apply [LMB, 3.8] to the tautological morphism $\pi\colon L_n \to \mathscr{B}_n$. By (1.6.4) and (1.7.5), π is locally essentially surjective (for the Zariski topology, hence for any finer topology). Moreover, it is clear from the definitions that the maps

$$\mathscr{A}ut(\mathbb{T}_n) \times L_n \underset{a}{\overset{\mathrm{pr}_{L_n}}{\rightrightarrows}} L_n,$$

where a denotes the action map described in (2.2.2), induce an isomorphism

$$\mathscr{A}ut(\mathbb{T}_n) \times L_n \xrightarrow{\sim} L_n \times_{\mathscr{B}_n} L_n.$$

The first assertion now follows, and, as in the proof of (2.1.2), the algebraicity assertion is immediate from [LMB, 4.6.1]. $\qquad\square$

Remark 2.3.3. As in (2.1.3), we deduce from the theorem that the quotient stack $\mathscr{A}ut(\mathbb{T}_n)\backslash L_n$ is independent of the topology on (Sch) between the Zariski and fpqc topologies, inclusive.

2.4. **Ind-infinitesimal sheaves.** To prepare for our discussion of the moduli stacks of formal Lie varieties and of formal Lie groups in the next two sections, it will be convenient to first dispense with a few generalities on ind-infinitesimal sheaves (1.4.5). Let S be a scheme, and recall the truncation functor

$$(\infty\text{-Inf})(S) \longrightarrow (n\text{-Inf})(S)$$

of (1.4.9). Since infinitesimal neighborhoods are compatible with base change (1.4.4), truncation preserves pullbacks (up to isomorphism), so that it specifies a morphism of fibered categories

$$(\infty\text{-Inf}) \longrightarrow (n\text{-Inf}).$$

Given $m \geq n$, the diagram

$$(\infty\text{-Inf}) \xrightarrow{\_^{(m)}} (m\text{-Inf}) \xrightarrow{\_^{(n)}} (n\text{-Inf})$$
$$\underbrace{\phantom{(\infty\text{-Inf}) \xrightarrow{\_^{(m)}} (m\text{-Inf}) \xrightarrow{\_^{(n)}}}}_{\_^{(n)}}$$

commutes up to canonical isomorphism (or possibly on the nose, depending on one's choice of definitions). Similarly, truncation $(n\text{-Inf}) \to (n\text{-Inf})$ is canonically isomorphic to the identity functor. Hence we may form the limit $\varprojlim_n (n\text{-Inf})$ of the fibered categories $(n\text{-Inf})$, $n \geq 0$, with respect to the truncation functors, and we obtain a natural arrow

$$(*) \qquad\qquad (\infty\text{-Inf}) \longrightarrow \varprojlim_n (n\text{-Inf}).$$

Let us emphasize that the limit $\varprojlim_n (n\text{-Inf})$ is taken in the sense of bicategories; see the appendix for a basic introduction, especially §A.6 for limits of fibered categories.

Proposition 2.4.1. *The arrow* $(*)$ *is an equivalence of fibered categories.*

Proof. Full faithfulness is an immediate consequence of the definition of ind-infinitesimal and the universal mapping property of a colimit. For essential surjectivity, recall (A.4.3, A.6.2) that an object in $\varprojlim_n (n\text{-Inf})$ over the scheme S is a family (X_n, φ_{mn}), where $X_n \in \mathrm{ob}(n\text{-Inf})(S)$ for all n, and $\varphi_{mn}\colon X_m^{(n)} \xrightarrow{\sim} X_n$ for all $m \geq n$,

subject to a natural cocycle condition for every $l \geq m \geq n$. For $m \geq n$, consider the composite

$$(**) \qquad\qquad X_n \xrightarrow[\sim]{\varphi_{mn}^{-1}} X_m^{(n)} \overset{\mathrm{can}}{\subset\!\longrightarrow} X_m.$$

As m and n vary, the cocycle condition on the φ_{mn}'s says that the various composites $(**)$ are organized into a diagram indexed on the totally ordered set $\mathbb{N} \cup \{0\}$. Let us take the colimit sheaf $X := \varinjlim_n X_n$. Since X is a colimit of a filtered diagram of sheaves, and all transition morphisms in the diagram are monomorphisms, the canonical arrow $X_n \to X$ is a monomorphism for all n. Since X_n is n-infinitesimal, $X_n \hookrightarrow X$ factors through $X^{(n)}$ for all n. It is now easy to verify that $X_n \hookrightarrow X^{(n)}$ is an isomorphism. Hence X is ind-infinitesimal. Moreover, it is easy to verify that the arrows $X_n \xrightarrow{\sim} X^{(n)}$, $n \geq 0$, define an isomorphism from (X_n, φ_{mn}) to the image of X in $\varprojlim_n (n\text{-Inf})$. $\qquad\square$

2.5. **The stack of formal Lie varieties.** We now come to the moduli stack of formal Lie varieties. Let S be a scheme.

Definition 2.5.1. We define $\mathscr{FLV}(S)$ to be the groupoid of formal Lie varieties and isomorphisms over S.

Remark 2.5.2. It is clear that the base change of a formal Lie variety is again a formal Lie variety. Hence $\mathscr{FLV}$ defines a CFG over (Sch). Moreover, it is clear from the definition of formal Lie variety (1.2.1) that $\mathscr{FLV}$ is a stack for the Zariski topology. In fact, $\mathscr{FLV}$ is a stack for the fpqc topology; this would not be hard to prove in a direct fashion, but we shall deduce it in (2.5.9) from the analogous statement for the stack of n-germs $\mathscr{G}_n$ (2.1.1).

Our first task for this section is to obtain the analog of (2.1.2) for $\mathscr{FLV}$. Recall the formal Lie variety $\widehat{\mathbb{A}}_S$ of (1.2.3).

Definition 2.5.3. We define $\mathscr{A}ut(\widehat{\mathbb{A}})$ to be the presheaf of groups on (Sch)

$$\mathscr{A}ut(\widehat{\mathbb{A}}) \colon S \longmapsto \mathrm{Aut}_{\mathscr{FLV}(S)}(\widehat{\mathbb{A}}_S).$$

We can describe $\mathscr{A}ut(\widehat{\mathbb{A}})$ in a way quite analogous to (1.6.6). Indeed, in analogy with $\mathbb{T}_{n,S}$, to give an automorphism of $\widehat{\mathbb{A}}_S$ is to give a power series $a_1 T + a_2 T^2 + \cdots \in \Gamma(S)[[T]]$ with a_1 a unit. So we deduce the following.

Proposition 2.5.4. *As a set-valued functor, $\mathscr{A}ut(\widehat{\mathbb{A}})$ is canonically represented by the open subscheme* $\mathrm{Spec}\, \mathbb{Z}[a_1, a_1^{-1}, a_2, a_3 \dots]$ *of* $\mathbb{A}_{\mathbb{Z}}^\infty$. $\qquad\square$

Theorem 2.5.5. $\mathscr{FLV} \approx B(\mathscr{A}ut(\widehat{\mathbb{A}}))$, *where the right-hand side denotes the classifying stack with respect to the Zariski topology.*

Proof. The proof is essentially the same as that of (2.1.2): $\mathscr{FLV}$ is plainly a gerbe over $\mathrm{Spec}\,\mathbb{Z}$ for the Zariski topology, and $\mathrm{Spec}\,\mathbb{Z} \xrightarrow{\widehat{\mathbb{A}}_{\mathbb{Z}}} \mathscr{FLV}$ defines a section. $\qquad\square$

Remark 2.5.6. Once we see in (2.5.9) that $\mathscr{FLV}$ is a stack for the fpqc topology, it will follow that $B(\mathscr{A}ut(\widehat{\mathbb{A}}))$ is independent of the choice of topology between the Zariski and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(\widehat{\mathbb{A}})$ is in fact a Zariski-torsor.

Remark 2.5.7. Unlike $\mathscr{G}_n$ (2.1.2), $\mathscr{FLV}$ is not algebraic: informally, the group $\mathscr{Aut}(\widehat{\mathbb{A}})$ is "too big", i.e. it is not of finite type.

Let us now turn to the relation between the stacks $\mathscr{FLV}$ and $\mathscr{G}_n$, $n \geq 0$. Recall that the truncation functors induce an equivalence $(\infty\text{-Inf}) \xrightarrow{\approx} \varprojlim_n (n\text{-Inf})$ (2.4.1). As noted in (1.5.4), this equivalence restricts to an arrow

$$(*) \qquad\qquad \mathscr{FLV} \longrightarrow \varprojlim_n \mathscr{G}_n.$$

As in the previous section, we emphasize that the limit in the display is taken in the sense of bicategories; see the appendix.

Theorem 2.5.8. *The arrow* $(*)$ *is an equivalence of stacks.*

Proof. Let (X_n, φ_{mn}) be an object in $\varprojlim_n \mathscr{G}_n$, say over the scheme S. As in the proof of (2.4.1), the φ_{mn}'s organize the X_n's into a diagram

$$(**) \qquad\qquad S \cong X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$$

indexed on the totally ordered set $\mathbb{N} \cup \{0\}$, such that the inclusion $X_n \hookrightarrow X_m$ identifies $X_n \xrightarrow[\sim]{\varphi_{mn}^{-1}} X_m^{(n)}$ for all $m \geq n$. In light of (2.4.1) and its proof, all we have to show is that the colimit sheaf $X := \varinjlim_n X_n$ is a formal Lie variety.

The proof is essentially the same as that of (1.6.4). Localizing (in the Zariski topology) on S, we may assume that S is an affine scheme $\operatorname{Spec} A$ and that X_1 has trivial conormal bundle. We claim that $X \simeq \operatorname{Spf} A[[T]]$. Indeed, by (1.6.2), we may assume that $X_n = \operatorname{Spec} B_n$, $n \geq 0$, for B_n an augmented A-algebra, say with augmentation ideal I_n satisfying $I_n^{n+1} = 0$. Then $(**)$ translates into a diagram of augmented A-algebras

$$A \cong B_0 \twoheadleftarrow B_1 \twoheadleftarrow B_2 \twoheadleftarrow \cdots$$

such that $B_m \twoheadrightarrow B_n$ identifies $B_m/I_m^{n+1} \xrightarrow{\sim} B_n$ for all $m \geq n$. Now, by assumption the conormal module I_1 (here $I_1^2 = 0$) for B_1 is a free A-module of rank 1. Let t_1 be any generator. Now successively choose a lift $t_2 \in B_2$ of t_1, a lift $t_3 \in B_3$ of t_2, and so on. Since $I_n/I_n^2 \cong I_1$ is free for all n, the proof of (1.6.4) shows exactly that the map

$$
\begin{aligned}
A[[T]] &\longrightarrow B_n \\
T &\longmapsto t_n
\end{aligned}
$$

induces $A[[T]]/(T)^{n+1} \xrightarrow{\sim} B_n$. Hence we obtain a topological isomorphism

$$A[[T]] \xrightarrow{\sim} \varprojlim_n B_n.$$

The theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 2.5.9. $\mathscr{FLV}$ *is a stack over* (Sch) *for the fpqc topology.*

Proof. Immediate from (2.1.1), (A.6.4), and the theorem. $\qquad\qquad\qquad\quad \square$

Remark 2.5.10. Our definition of formal Lie variety (1.2.1) has a kind of built-in local triviality for the Zariski topology. Though one may be tempted to consider formulating the local triviality condition with respect to other topologies, the corollary says that the notion of formal Lie variety is independent of the choice of topology for local triviality between the Zariski and fpqc topologies, inclusive.

2.6. **The stack of formal Lie groups.** Now that we have discussed the stack of formal Lie varieties, we turn to the moduli stack of formal Lie groups. Let S be a scheme.

Definition 2.6.1. We define (FLG)(S) to be the category of formal Lie groups and homomorphisms over S. We define $\mathscr{FLG}(S)$ to be the groupoid of formal Lie groups and isomorphisms over S.

Remark 2.6.2. Since formal Lie varieties are stable under base change, it is clear that formal Lie groups are stable under base change. Hence (FLG) defines a fibered category, and $\mathscr{FLG}$ a CFG, over (Sch). We shall verify in (2.6.6) that $\mathscr{FLG}$ is a stack for the fpqc topology. In some sense, this almost follows in a purely formal fashion from the analogous result for $\mathscr{FLV}$ (2.5.9). The only difficulties arise when trying to glue locally given multiplication maps $X \times_S X \to X$: first, strictly speaking, the product of formal Lie varieties is not a (1-parameter) formal Lie variety, hence not an object of $\mathscr{FLV}$; and second, $\mathscr{FLV}$ only parametrizes *isomorphisms* between objects, not arbitrary morphisms. Of course, these difficulties are merely artifacts of our choices to work only with 1-parameter formal Lie varieties and only with stacks of groupoids. We would have no trouble if we had worked from the outset with pointed formal Lie varieties in the *general* sense of (1.2.2), with no constraint on the number of parameters, and with stacks of *categories*.

In analogy with (1.7.11), given a group law F over $\Gamma(S)$ and a base change $f\colon S' \to S$, one has $\widehat{\mathbb{A}}_S^F \times_S S' \cong \widehat{\mathbb{A}}_{S'}^{F'}$, where F' is the group law over $\Gamma(S')$ obtained by applying $f^{\#}$ to the coefficients of F.

Our first goal in this section is to prove the analog of (2.3.2) for $\mathscr{FLG}$. Recall the formal Lie variety $\widehat{\mathbb{A}}_S$ of (1.2.3).

Definition 2.6.3. We define L to be the presheaf of sets on (Sch)

$$L\colon S \longmapsto \{\text{formal Lie group structures on } \widehat{\mathbb{A}}_S\}.$$

In analogy with L_n (2.2.1), we have $L \simeq \operatorname{Spec} \mathbb{Z}[a_1, a_2, \dots] = \mathbb{A}_{\mathbb{Z}}^{\infty}$ by Lazard's theorem (1.1.7), but the isomorphism is not canonical. Just as in (2.2.2), $\mathscr{A}ut(\widehat{\mathbb{A}})$ acts naturally on L as "changes of coordinate". Just as in (2.3.2), we deduce the following.

Theorem 2.6.4. $\mathscr{FLG} \approx \mathscr{A}ut(\widehat{\mathbb{A}})\backslash L$. $\qquad\square$

Remark 2.6.5. It is an immediate consequence of (2.5.6) that the quotient stack $\mathscr{A}ut(\widehat{\mathbb{A}})\backslash L$ is independent of the choice of topology between the Zariski and fpqc topologies, inclusive.

Corollary 2.6.6. $\mathscr{FLG}$ *is a stack for the fpqc topology.* $\qquad\square$

Remark 2.6.7. In analogy with (2.5.7), unlike the moduli stack of n-buds $\mathscr{B}_n$ (2.3.2), $\mathscr{FLG}$ is not algebraic.

In analogy with the previous section, let us now turn to the relation between the stacks $\mathscr{FLG}$ and $\mathscr{B}_n$, $n \geq 0$. By (1.7.7), we may form the limit $\varprojlim_n \mathscr{B}_n$ of the $\mathscr{B}_n$'s with respect to the truncation functors, and we obtain an arrow

$$(*) \qquad\qquad \mathscr{FLG} \longrightarrow \varprojlim_n \mathscr{B}_n.$$

As in previous sections, we emphasize that the limit is taken in the sense of bicategories; see the appendix.

Theorem 2.6.8. *The arrow* $(*)$ *is an equivalence of stacks.*

Proof. We could prove the theorem in a fairly direct fashion entirely analogous to (2.5.8). Instead, we can obtain the theorem as a more-or-less formal consequence of (2.5.8). Indeed, for any base scheme S and $n \geq 0$, the category $(n\text{-Inf})(S)$ contains finite products (1.4.10), and truncation between the $(n\text{-Inf})(S)$'s for varying n preserves finite products. Hence passing to the limit of the $(n\text{-Inf})(S)$'s commutes with taking commutative group objects (A.7.2). Now use (2.5.8) and the identification of $\mathscr{B}_n(S)$ with objects of $\mathscr{G}_n(S)$ endowed with a commutative group structure (1.8.1). $\qquad\square$

3. The height stratification: buds

Fix a prime number p once and for all. We shall now begin to study the algebraic geometry of the classical notion of *height* for formal group laws and bud laws over rings of characteristic p. The essential feature of the theory is a *stratification*, relative to p, on the moduli stack $\mathscr{FLG}$ of formal Lie groups and on the stacks $\mathscr{B}_n$ of n-buds. We shall begin by working in the context of buds; we shall turn to formal Lie groups in §4.

In order to reduce clutter, we shall choose not to embed p in our notation, though one certainly obtains a different stratification for each choice of p.

3.1. **Review of formal group laws II.** We devote this section to a brief survey of some of the classical algebraic theory of formal group laws and bud laws related to height, as well as some somewhat technical refinements to the material in §1.1. We shall use [L] and [F] as our main references, although there are many other sources available. As for §1.1, the reader may wish to skip this section and refer back only as needed.

Let A be a ring and F an n-bud law, $n \geq 1$, or a formal group law.

Definition 3.1.1. For $m \in \mathbb{N}$, we define $[m]_F$ to be the element of $A[T]/(T)^{n+1}$ in the bud law case, and of $A[[T]]$ in the group law case,

$$[m]_F(T) := F(\cdots F(F(\underbrace{T, T), T), \cdots, T)}_{m\ T\text{'s}}.$$

In other words, $[1]_F(T) = T$, and $[m+1]_F(T) = F\big([m]_F(T), T\big)$.

Remark 3.1.2. Since

$$F(T_1, T_2) = T_1 + T_2 + (\text{higher order terms}),$$

we have

$$[m]_F(T) = mT + (\text{higher order terms}).$$

We will be almost exclusively interested in $[p]_F$ for varying bud or group laws F. If A is of characteristic p, then we conclude from (3.1.2) that $[p]_F = 0$ or has lowest degree term of degree ≥ 2. In fact, a much stronger statement holds.

Proposition 3.1.3. *Suppose A has characteristic p. Let $f\colon F \to G$ be a homomorphism of bud laws or formal group laws over A. Then f is 0 or of the form $a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$ for some nonnegative integer h and some nonzero $a_{p^h} \in A$. In particular, $[p]_F$ is 0 or of the form $a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$, $a_{p^h} \neq 0$.*

Proof. [F, I §3 Theorem 2(ii)], which works for bud laws as well as formal group laws. Note that $[p]_F$ is plainly a homomorphism $F \to F$. $\square$

The proposition motivates the following definition. Let h be a nonnegative integer.

Definition 3.1.4. F has *height h* if $[p]_F$ is of the form $a_{p^h} T^{p^h} + (\text{higher order terms})$ for some $a_{p^h} \in A^\times$.

Remark 3.1.5. In the bud law case, height h only makes sense for $n \geq p^h$.

Remark 3.1.6. By (3.1.2), bud laws and group laws of positive height only occur over rings of characteristic p. If F is of positive height h, then F is a truncated polynomial in the bud law case, and power series in the group law case, in T^{p^h} (3.1.3). At the other extreme, F has height $0 \iff A$ is a $\mathbb{Z}[\frac{1}{p}]$-algebra.

Remark 3.1.7. The definition of height in (3.1.4) differs from the one in the classical literature over rings of characteristic p; compare e.g. [L, p. 266] or [F, p. 27]. Explicitly, F has height h in the classical sense if

$$[p]_F(T) = a_{p^h} T^{p^h} + (\text{higher order terms})$$

with $a_{p^h} \neq 0$, whereas we require, in addition, that a_{p^h} be a *unit*. In the classical sense, every F is of finite height or satisfies $[p]_F = 0$. But in the sense of (3.1.4), there certainly exist F for which $[p]_F \neq 0$ but the height is not defined. The modified version of the classical definition allows one to obtain better behavior with regard to change of base ring.

Remark 3.1.8. In the group law case, it is customary to say that F is of *infinite height* if $[p]_F = 0$. For example, over a ring of characteristic p, the additive law $F(T_1, T_2) = T_1 + T_2$ has infinite height. We remark that this notion would fit naturally into the geometric theory of formal Lie groups, but we will not have occasion to introduce it here.

Before continuing with our discussion of $[p]$, it is now convenient to digress for a moment on some technical matters underpinning some of the results we surveyed in §1.1. Quite generally, let $P \in A[T_1, T_2]$ be any polynomial.

Definition 3.1.9. We say that P is a *symmetric polynomial 2-cocycle*, or *SPC* for short, if

(S) $P(T_1, T_2) = P(T_2, T_1)$ in $A[T_1, T_2]$; and
(C) $P(T_2, T_3) - P(T_1 + T_2, T_3) + P(T_1, T_2 + T_3) - P(T_1, T_2) = 0$ in $A[T_1, T_2, T_3]$.

Remark 3.1.10. To explain the terminology, recall that the group cohomology $H^\bullet(A, A)$ can be computed by the cochain complex $\left(C^\bullet(A, A), \delta^\bullet\right)$, where $C^i := C^i(A, A) := \mathrm{Hom}_{(\mathrm{Sets})}(A^i, A)$, $i \geq 0$, and $\delta^i : C^i \to C^{i+1}$ sends

$$f(a_1, \ldots, a_i) \overset{\delta^i}{\longmapsto} f(a_2, \ldots, a_{i+1})$$
$$+ \sum_{j=1}^{i} (-1)^j f(a_1, \ldots, a_{j-1}, a_j + a_{j+1}, a_{j+2}, \ldots, a_{i+1})$$
$$+ (-1)^{i+1} f(a_1, \ldots, a_i).$$

For each i, the polynomial ring $A[T_1, \ldots, T_i]$ maps in the obvious way to C^i. Hence (C) in the definition justifies "polynomial 2-cocycle". Plainly (S) justifes "symmetric".

Example 3.1.11. For $m \geq 2$, let

$$\lambda(m) := \begin{cases} l, & l \text{ is a prime and } m \text{ is a power of } l; \\ 1, & \text{otherwise.} \end{cases}$$

Then we define Lazard's polynomials $B_m \in \mathbb{Z}[T_1, T_2]$ by

$$B_m(T_1, T_2) := (T_1 + T_2)^m - T_1^m - T_2^m$$

and $C_m \in \mathbb{Z}[T_1, T_2]$ by

$$C_m(T_1, T_2) := \frac{1}{\lambda(m)} B_m(T_1, T_2).$$

Since $\delta^1(-T^m) = B_m$, we deduce at once that B_m and C_m are SPC's over $\mathbb{Z}$, homogenous of degree m. Hence B_m and C_m define SPC's over any ring.

Note that over a ring of characteristic p and for m a power of p, we have $B_m = 0$. But the SPC C_m is more interesting. The following is one of Lazard's most important technical results, valid for any ring A and for any $m \geq 2$.

Theorem 3.1.12 (Lazard). *Let P be a homogenous SPC of degree m over A. Then there exists a unique $a \in A$ such that $P = aC_m$.*

Proof. It is easy to compute directly that C_m is *primitive*, which implies the unicity of a. Existence of a is much harder; see [L, Lemme 3] or [F, III §1 Theorem 1a]. □

SPC's occur in the theory of formal group laws and bud laws in the following simple way; we shall encounter them in a somewhat different context in §3.7. Let $m \geq 2$, and let F and G be group laws or n-bud laws over A whose respective $(m-1)$-truncations $F^{(m-1)}$ and $G^{(m-1)}$ (1.7.7) are equal, $m \leq n-1$ in the bud case. Then a fairly simple calculation shows that $F^{(m)}$ and $G^{(m)}$ differ by a homogenous SPC of degree m. Hence (3.1.12) implies the following.

Corollary 3.1.13 (Lazard). *There exists a unique $a \in A$ such that*

$$F^{(m)} = G^{(m)} + aC_m. \qquad \qquad \square$$

The corollary plays an important role in Lazard's construction of a universal group law U over $\mathbb{Z}[t_1, t_2, \ldots]$ we cited in (1.1.7). We will use the following refinement of (1.1.7) for calculations in §3.5. It describes the group law constructed by Lazard in the proofs of [L, Théorèmes II and III].

Theorem 3.1.14 (Lazard). *The universal formal group law $U(T_1, T_2)$ of (1.1.7) may be chosen such that for all $n \geq 1$,*

  (i) *the coefficients of the truncation $U^{(n)}$ involve only $t_1, \ldots, t_{n-1}$;*
  (ii) *$U^{(n)}$, regarded as defined over $\mathbb{Z}[t_1, \ldots, t_{n-1}]$ by (i), is a universal n-bud law; and*
  (iii) *there is an equality of elements in $\mathbb{Z}[t_1, \ldots, t_{n-1}, s][T_1, T_2]/(T_1, T_2)^{n+1}$,*

$$U^{(n)}(t_1, \ldots, t_{n-2}, t_{n-1} + s)(T_1, T_2) - U^{(n)}(t_1, \ldots, t_{n-1},)(T_1, T_2)$$

$$= sC_n(T_1, T_2). \quad \square$$

We now return to our discussion of $[p]$. We will use the following lemma to study the coefficients of $[p]$ in §3.5. Let F, G, and m be as in (3.1.13), so that $F^{(m)} = G^{(m)} + aC_m$ for a unique $a \in A$.

Lemma 3.1.15. *For any $k \geq 1$,*

$$[k]_{F^{(m)}}(T) = [k]_{G^{(m)}}(T) + \frac{k^m - k}{\lambda(m)} \cdot aT^m.$$

In particular, for $k = p$ and m of the form p^h, we have

$$[p]_{F^{(p^h)}}(T) = [p]_{G^{(p^h)}}(T) + \left(p^{p^h-1} - 1\right)aT^{p^h}.$$

Proof. [L, Lemme 6] or [F, III §1 Lemma 4]. $\qquad\qquad\square$

In later sections, we will be especially interested in group laws of a given height for which $[p]$ is as simple as possible. Fix $h \geq 1$.

Theorem 3.1.16. *There exists a formal group law H_h over $\mathbb{F}_p$ such that $[p]_{H_h}(T) = T^{p^h}$.*

Proof. See, for example, [F, III §2 Theorem 1]. $\qquad\qquad\square$

Notation 3.1.17. We fix once an for all a formal group law H_h as in (3.1.16). As we shall often work with respect to fixed h, we often write just H for H_h, provided no confusion seems likely.

If A is of characteristic p, then we may view H as defined over A. Let $n \geq p^h$, and consider the n-truncation $H^{(n)}$.

Proposition 3.1.18. *Let F be an n-bud law (resp., formal group law) of height h over A. Then there exists a finite étale extension ring (resp., a countable ascending union of finite étale extension rings) B of A such that $F \simeq H^{(n)}$ (resp., $F \simeq H$) over B.*

Proof. We'll proceed by extracting some arguments from the proofs of the statements leading up to the proof of Theorem 2 in [F, III §2]. There is also a somewhat cleaner version of the proof sketched in [Mi, 10.4].

It suffices to consider the bud law case; the group law case then follows by considering the various truncations $F^{(n)}$ for higher and higher n. To begin, the proof of [F, III §2 Lemma 3] shows that there exists a finite étale extension ring B of A such that F is isomorphic over B to an n-bud law G for which $[p]_G(T) = T^{p^h}$. A bit more explicitly, whereas [F] takes A to be a separably closed field and proceeds by finding solutions to certain (separable) equations in A, one can proceed in our case by formally adjoining solutions to certain (separable) equations to A, that is, one can obain B as an iterated extension ring of (finitely many) rings of the form $A[X]$ modulo a separable polynomial.

The remaining step is to show that over any ring, any two bud laws for which $[p](T) = T^{p^h}$ are isomorphic. This is probably best and most simply seen via a direct argument, but it can be gleaned from [F, III §2] by combining arguments (suitably adapted to the bud case) in the proofs of Lemma 2, Proposition 3, and Theorem 2. $\qquad\qquad\square$

Remark 3.1.19. In particular, the ring B in the theorem is faithfully flat over A.

3.2. **Multiplication by p.** Let X be an n-bud or formal Lie group over the scheme S. Then (using (1.8.2) in the bud case) the endomorphisms of X form an abelian group. So we may make the following definition.

Definition 3.2.1. We define $[p]_X$ to be the element of $\mathrm{End}_{(n\text{-buds})(S)}(X)$ in the bud case, and of $\mathrm{End}_{(\mathrm{FLG})(S)}(X)$ in the formal Lie group case, $p\,\mathrm{id}_X$.

Remark 3.2.2. When $X = \mathbb{T}_S^F$ for an n-bud law $F \in \Gamma_n(S; T_1, T_2)$ (1.7.5), the morphism
$$X \xrightarrow{\;[p]_X\;} X$$
corresponds to the $\mathscr{O}_S$-algebra map
$$\mathscr{O}_S[T]/(T)^{n+1} \longleftarrow \mathscr{O}_S[T]/(T)^{n+1}$$
$$[p]_F(T) \longleftarrow\!\!\!\longmapsto T,$$
with $[p]_F$ as defined in (3.1.1). There is an obvious analogous statement when F is a group law.

Remark 3.2.3. One checks at once that truncation functors are additive functors on the category of formal Lie groups and on the various categories of buds. Hence truncation preserves $[p]$: if X is a formal Lie group or an m-bud with $m \geq n$, then $[p]_X^{(n)} = [p]_{X^{(n)}}$.

Remark 3.2.4. Similarly, consider the category of n-buds or of formal Lie groups over S. Then for any morphism $f\colon S' \to S$, the base change functor $-\times_S S'$ is additive, hence preserves $[p]$. When $X = \mathbb{T}_S^F$ or $X = \widehat{\mathbb{A}}_S^F$, recall that the multiplication law on $X \times_S S'$ is given by the law F' obtained by applying $f^\#$ to the coefficients of F (1.7.11, 2.6.2). Hence $[p]_{F'}$ is obtained by applying $f^\#$ to the coefficients of $[p]_F$.

3.3. **Zero loci of line bundles.** In the next section we'll describe the height stratification on $\mathscr{B}_n$ as arising from a succession of *zero loci of sections of line bundles*. Our aim in this section is to dispense with a few of the basic preliminaries. The material we shall discuss is general in nature and is independent of our earlier work.

Let (Vect_1) denote the fibered category on (Sch) that assigns to each scheme S the category of locally free $\mathscr{O}_S$-modules of rank 1 and all module homomorphisms (with the usual pullback functors). Then (Vect_1) is an fpqc stack [SGA1, VIII 1.12]. Note that the underlying moduli stack of (Vect_1), obtained by discarding the non-Cartesian morphisms, is just $B(\mathbb{G}_m)$. Let $\mathscr{F}$ be a fibered category over (Sch).

Definition 3.3.1. A *line bundle* on $\mathscr{F}$ is a 1-morphism $\mathscr{L}\colon \mathscr{F} \to (\mathrm{Vect}_1)$ in $\mathrm{Fib}\big((\mathrm{Sch})\big)$. A *morphism* $\mathscr{L} \to \mathscr{L}'$ of line bundles on $\mathscr{F}$ is a 2-morphism $\mathscr{L} \to \mathscr{L}'$ in $\mathrm{Fib}\big((\mathrm{Sch})\big)$.

When $\mathscr{F}$ is an algebraic stack, one recovers the usual notion of line bundle on $\mathscr{F}$; see [LMB, 13.3] (strictly speaking, [LMB] would take $(\mathrm{Vect}_1)(S)$ to be the *opposite* of the category of locally free $\mathscr{O}_S$-modules of rank 1, but let us not belabor this point).

Example 3.3.2. For any fibered category $\mathscr{F}$, we denote by $\mathscr{O}_\mathscr{F}$ the line bundle that assigns to each $X \in \mathrm{ob}\,\mathscr{F}$ over the scheme S the trivial line bundle $\mathscr{O}_S$, and

to each morphism $\mu\colon Y \to X$ over $f\colon T \to S$ the Cartesian morphism in (Vect_1) corresponding to the canonical isomorphism $\mathscr{O}_T \xrightarrow{\sim} f^*\mathscr{O}_S$.

Let $\mathscr{L}$ be a line bundle on the fibered category $\mathscr{F}$.

Definition 3.3.3.
  (a) A *global section*, or just *section, of $\mathscr{L}$* is a morphism $a\colon \mathscr{O}_{\mathscr{F}} \to \mathscr{L}$.
  (b) Given a section a of $\mathscr{L}$, the *zero locus of a* is the full subcategory $V(a)$ of $\mathscr{F}$ whose objects X over the scheme S are those for which $a_X\colon \mathscr{O}_S \to \mathscr{L}_X$ is the 0 map.

Let a be a section of $\mathscr{L}$. The basic result of interest to us is the following. The proof is straightforward.

Proposition 3.3.4.
  (i) $V(a)$ *is a sub-fibered category of $\mathscr{F}$, and the inclusion functor $V(a) \to \mathscr{F}$ is a closed immersion.*
  (ii) *If $\mathscr{F}$ is a CFG, or stack, or algebraic stack, then so is $V(a)$.* $\square$

3.4. **The height stratification on the stack of n-buds.** In this section, we translate the classical notion of height for bud laws to the geometric setting by defining the *height stratification* on the stack $\mathscr{B}_n$. We shall define the analogous stratification on the stack of formal Lie groups in §4.1.

Let X be an n-bud over the scheme S with section $\sigma\colon S \to X$. Let

$$\mathscr{I}_X := \ker\!\big[\sigma^{-1}\mathscr{O}_X \xrightarrow{\sigma^\sharp} \mathscr{O}_S\big].$$

Then the endomorphism $[p]_X$ of X (3.2.1) determines an endomorphism $[p]_{\mathscr{I}_X}$ of $\mathscr{I}_X$.

Now let h be a nonnegative integer, and assume $n \geq p^h$.

Definition 3.4.1. We say X has *height $\geq h$* if the endomorphism $[p]_{\mathscr{I}_X}\colon \mathscr{I}_X \to \mathscr{I}_X$ has image in $\mathscr{I}_X^{p^h}$. We denote by $\mathscr{B}_n^{\geq h}$ the full subcategory of $\mathscr{B}_n$ of n-buds of height $\geq h$.

Example 3.4.2. Let $X = \mathbb{T}_S^F$ for some n-bud law F over $\Gamma(S)$ (1.7.5). Then we see from (3.2.2) that X has height $\geq h \iff [p]_F \in T^{p^h} \cdot \Gamma_n(S;T)$. So our terminology is compatible with (3.1.4).

Remark 3.4.3. Recall that $[p]$ (3.2.4) is compatible with base change. Hence height $\geq h$ is stable under base change. Hence $\mathscr{B}_n^{\geq h}$ is fibered over (Sch).

Remark 3.4.4. Similarly, height $\geq h$ is stable under truncation (provided we don't truncate below $n = p^h$). More precisely, X has height $\geq h \iff X^{(p^h)}$ has height $\geq h$.

Remark 3.4.5. Of course, for fixed n, there are only finitely many values (possibly none) of h for which height $\geq h$ makes sense. So we get a finite decreasing chain $\mathscr{B}_n = \mathscr{B}_n^{\geq 0} \supsetneq \mathscr{B}_n^{\geq 1} \supsetneq \mathscr{B}_n^{\geq 2} \supsetneq \cdots$.

We shall see in a moment that the inclusion $\mathscr{B}_n^{\geq h} \hookrightarrow \mathscr{B}_n$ is a *closed immersion*. First, some notation.

By definition of n-germ (1.5.1) for $n \geq 1$, the conormal sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ associated to the section $S \to X$ is a line bundle on S. Moreover, since the conormal sheaf

associated to a section is compatible with base change on S [EGAIV$_4$, 16.2.3(ii)], formation of the conormal bundle defines a line bundle on $\mathscr{B}_n$.

Definition 3.4.6. We denote by $\mathscr{L}$ the line bundle on $\mathscr{B}_n$ associating to each bud its associated conormal sheaf. We define $\mathscr{L}_h := \mathscr{L}|_{\mathscr{B}_n^{\geq h}}$.

Remark 3.4.7. Clearly, the same construction defines a line bundle $\mathscr{L}'$ on the stack $\mathscr{G}_n$ of n-germs, and $\mathscr{L}$ is just the pullback of $\mathscr{L}'$ along the natural forgetful morphism $\mathscr{B}_n \to \mathscr{G}_n$.

Remark 3.4.8. Strictly speaking, $\mathscr{L}$ and $\mathscr{L}_h$ depend on n. But the dependence on n is largely superficial: since the conormal sheaf of an immersion depends only on the 1st infinitesimal neighborhood, formation of $\mathscr{L}$ is compatible with truncation $\mathscr{B}_m \to \mathscr{B}_n$, and similarly for $\mathscr{L}_h$. In other words, up to canonical isomorphism, we may construct $\mathscr{L}$ on $\mathscr{B}_n$ by first constructing $\mathscr{L}$ on $\mathscr{B}_1$ and then pulling back along $\mathscr{B}_n \to \mathscr{B}_1$; and similarly for $\mathscr{L}_h$, replacing $\mathscr{B}_1$ by $\mathscr{B}_{p^h}$. So, to avoid clutter, we shall abuse notation and simply suppress the n when writing $\mathscr{L}$ and $\mathscr{L}_h$.

When X has height $\geq h$, the map $[p]_{\mathscr{I}_X} : \mathscr{I}_X \to \mathscr{I}_X^{p^h}$ induces a map of line bundles

$$(*) \qquad \mathscr{I}_X / \mathscr{I}_X^2 \longrightarrow \mathscr{I}_X^{p^h} / \mathscr{I}_X^{p^h+1}.$$

But $\mathscr{I}_X^{p^h} / \mathscr{I}_X^{p^h+1} \cong (\mathscr{I}_X / \mathscr{I}_X^2)^{\otimes p^h}$ (1.6.3). So we may express $(*)$ as

$$(\mathscr{L}_h)_X \longrightarrow (\mathscr{L}_h)_X^{\otimes p^h},$$

or as a section

$$(**) \qquad (v_h)_X : \mathscr{O}_S \longrightarrow (\mathscr{L}_h)_X^{\otimes p^h - 1}.$$

Since $[p]$ is compatible with pullbacks, we may make the following definition.

Definition 3.4.9. We denote by v_h the section of $\mathscr{L}_h^{\otimes p^h - 1}$ defined by $(**)$.

Example 3.4.10. By (3.1.2), v_0 is just the section p of $\mathscr{O}_{\mathscr{B}_n}$.

Example 3.4.11. Of course, we've taken pains to express v_h in a coordinate-free way, so that it is, in some sense, canonical. But when X admits a coordinate, $\mathscr{L}_X$ is trivial, and v_h can be understood more concretely. Precisely, suppose $X = \mathbb{T}_S^F$ for some n-bud law F over $\Gamma(S)$. Then $\mathscr{I}_X = T \cdot \mathscr{O}_S[T]/(T)^{n+1}$, and there is an obvious trivialization

$$\mathscr{O}_S \xrightarrow{\sim} \mathscr{I}_X / \mathscr{I}_X^2 = \mathscr{L}_X$$
$$1 \longmapsto \text{image of } T.$$

The displayed trivialization induces a natural trivialization of $\mathscr{L}_X^{\otimes p^h - 1}$.

Now suppose X has height $\geq h$, so that $[p]_F$ is of the form

$$a_{p^h} T^{p^h} + (\text{higher order terms}) \qquad (3.4.2).$$

Then, under our trivialization of $(\mathscr{L}_h)_X^{\otimes p^h - 1} = \mathscr{L}_X^{\otimes p^h - 1}$, $(v_h)_X$ corresponds exactly to $a_{p^h} \in \Gamma(S)$.

Remark 3.4.12. Just as for $\mathscr{L}_h$, v_h implicitly depends on n. But v_h is essentially independent of n in the same sense as $\mathscr{L}_h$ is (3.4.8).

Proposition 3.4.13. *Assume $h \geq 1$. Then $\mathscr{B}_n^{\geq h}$ is the zero locus (3.3.3) in $\mathscr{B}_n^{\geq h-1}$ of the section v_{h-1}.*

Proof. Let X be an n-bud of height $\geq h-1$. We claim

$$X \in \operatorname{ob} V(v_{h-1}) \iff (v_{h-1})_X = 0$$

$$\iff [p]_{\mathscr{I}_X} \text{ carries } \mathscr{I}_X \text{ into } \mathscr{I}_X^{p^{h-1}+1}$$

$$\overset{(\dagger)}{\iff} [p]_{\mathscr{I}_X} \text{ carries } \mathscr{I}_X \text{ into } \mathscr{I}_X^{p^h}$$

$$\iff X \in \operatorname{ob} \mathscr{B}_n^{\geq h}.$$

Only the "$\iff$" marked ($\dagger$) requires proof. For this, the implication $\impliedby$ is trivial. For the implication $\implies$, the assertion is local on S, so we may assume X is of the form $\mathbb{T}_S^F$ (1.7.5). Now use (3.1.3) and (3.2.2). $\square$

Corollary 3.4.14. *$\mathscr{B}_n^{\geq h}$ is an algebraic stack for the fpqc toplogy, and the inclusion functor $\mathscr{B}_n^{\geq h} \to \mathscr{B}_n$ is a closed immersion.*

Proof. Apply (2.3.1), (2.3.2), (3.3.4), and the proposition. $\square$

Remark 3.4.15. Combining (3.4.10) and (3.4.13), we see that $\mathscr{B}_n^{\geq 1}$ is precisely the stack of n-buds over $\mathbb{F}_p$-schemes.

Remark 3.4.16. The proposition says that the property of height $\geq h$ depends only on a bud's p^{h-1}-truncation. So we could extend the notion of height $\geq h$ to n-buds for $n \geq p^{h-1}$, but this added bit of generality would offer no real advantage to us.

3.5. **The stack of height $\geq h$ n-buds.** Let $h \geq 1$ and $n \geq p^h$. Our aim in this section is to obtain a description of $\mathscr{B}_n^{\geq h}$ (3.4.1) analogous to the description of $\mathscr{B}_n$ in (2.3.2).

As a warm-up, note first that the case $h = 1$ is easy: $\mathscr{B}_n^{\geq 1} \approx \mathscr{B}_n \otimes \mathbb{F}_p$ by (3.4.15), so $\mathscr{B}_n^{\geq 1} \approx \mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \backslash (L_n)_{\mathbb{F}_p}$ by (2.3.2), where $(L_n)_{\mathbb{F}_p} := L_n \otimes \mathbb{F}_p \simeq \mathbb{A}_{\mathbb{F}_p}^{n-1}$ and $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} := \mathscr{A}ut(\mathbb{T}_n) \otimes \mathbb{F}_p$ is an open subscheme of $\mathbb{A}_{\mathbb{F}_p}^n$.

To treat the case of general h, let us begin by constructing a convenient presentation of $\mathscr{B}_n^{\geq h}$. Let A be the polynomial ring $\mathbb{Z}_{(p)}[t_1, \ldots, t_{n-1}]$, and let F be a universal (for $\mathbb{Z}_{(p)}$-algebras) n-bud law over A such that the truncation $F^{(n')}$ satisfies (i)–(iii) in (3.1.14) for all $n' \leq n$. Let a_h denote the coefficient of T^{p^h} in $[p]_F(T) \in A[T]/(T)^{n+1}$ (3.1.1). For $h \geq 1$, we define

- $A_h := A/(p, a_1, \ldots, a_{h-1})$;
- $F_h :=$ the reduction of F over A_h; and
- $X_h := \mathbb{T}_{A_h}^{F_h}$.

By (3.1.3) and (3.4.2), X_h has height $\geq h$. So X_h determines (up to isomorphism) a classifying map

$$(*) \qquad\qquad \operatorname{Spec} A_h \xrightarrow{X_h} \mathscr{B}_n^{\geq h}.$$

Theorem 3.5.1. *$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ acts naturally on $\operatorname{Spec} A_h$, and $(*)$ identifies*

$$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \backslash \operatorname{Spec} A_h \approx \mathscr{B}_n^{\geq h}.$$

Proof. The theorem is clear for $h = 1$, since plainly X_1 identifies $\operatorname{Spec} A_1 \xrightarrow{\sim} (L_n)_{\mathbb{F}_p}$. Now, for any h, the theorem is equivalent by [LMB, 3.8] to

- ($*$) is locally essentially surjective;
- $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ acts on $\operatorname{Spec} A_h$; and
- the $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$-action induces

$$\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \times \operatorname{Spec} A_h \cong \operatorname{Spec} A_h \times_{\mathscr{B}_n^{\geq h}} \operatorname{Spec} A_h.$$

So we need note just that the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} A_{h+1} & \hookrightarrow & \operatorname{Spec} A_h \\
{\scriptstyle X_{h+1}}\downarrow & & \downarrow{\scriptstyle X_h} \\
\mathscr{B}_n^{\geq h+1} & \hookrightarrow & \mathscr{B}_n^{\geq h}
\end{array}
$$

is Cartesian by (3.4.11) and (3.4.13), and everything follows by induction on h. $\quad\square$

Remark 3.5.2. Though we didn't need it explicitly in the proof of the theorem, the ring A_h and the bud law F_h admit an obvious modular interpretation: namely, F_h is *universal* amongst n-bud buds of height $\geq h$. On the other hand, one could use universality of F_h to give an alternative proof of the theorem quite along the lines of (2.3.2), without appealing to induction.

The theorem leads us to consider closely the ring A_h. The essential observation is the following result on A.

Proposition 3.5.3. *The map of polynomial rings $\mathbb{Z}_{(p)}[u_1,\ldots,u_{n-1}] \to A$ determined by*

$$(**)\qquad\qquad u_i \longmapsto \begin{cases} a_h & i = p^h - 1 \text{ for } h = 1,\, 2,\,\ldots; \\ t_i & otherwise \end{cases}$$

is an isomorphism. In particular, $A/(a_1,\ldots,a_{h-1})$ is a polynomial ring over $\mathbb{Z}_{(p)}$ on $n - h$ variables.

Proof. Combining (3.1.15) with the form of F described in (3.1.14), we deduce that

$$a_h(t_1,\ldots,t_{p^j-2}, t_{p^h-1} + s) - a_h(t_1,\ldots,t_{p^j-1}) = (p^{p^h-1} - 1)s$$

as elements in $\mathbb{Z}_{(p)}[t_1,\ldots,t_{p^h-1}, s]$. Hence

$$a_h(t_1,\ldots,t_{p^h-1}) = (p^{p^h-1} - 1)t_{p^h-1} + (\text{terms involving } t_1,\ldots,t_{p^h-2}).$$

But, for $h \geq 1$, $p^{p^h-1} - 1$ is a *unit* in any $\mathbb{Z}_{(p)}$-algebra. The proposition now follows easily. $\quad\square$

Corollary 3.5.4. $\operatorname{Spec} A_h \simeq \mathbb{A}_{\mathbb{F}_p}^{n-h}$. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.5.5. There are natural analogs of (3.5.3) and (3.5.4) in the group law setting: if U is a universal (for $\mathbb{Z}_{(p)}$-algebras) formal group law over $\mathbb{Z}_{(p)}[t_1, t_2,\ldots]$ of the form described in (3.1.14), and we again denote by a_h the coefficient of T^{p^h} in $[p]_U(T)$, then

- the map $\mathbb{Z}_{(p)}[u_1, u_2,\ldots] \to \mathbb{Z}_{(p)}[t_1, t_2,\ldots]$ specified by ($**$) is an isomorphism of polynomial rings; and
- $\mathbb{Z}_{(p)}[t_1, t_2,\ldots]/(a_1,\ldots,a_{h-1})$ is a polynomial ring over $\mathbb{Z}_{(p)}$ on the images of the t_i for $i \neq p^1 - 1,\, p^2 - 1,\,\ldots,\, p^{h-1} - 1$.

Moreover, in analogy with (3.5.2), the reduction of U over

$$\mathbb{Z}_{(p)}[t_1, t_2, \dots]/(p, a_1, \dots, a_{h-1}) \cong \mathbb{F}_p[t_1, t_2, \dots]/(\overline{a}_1, \dots, \overline{a}_{h-1}),$$

where $\overline{a}_i$ denotes the reduction of a_i mod p, is plainly of height $\geq h$, and indeed *universal* amongst group laws of height $\geq h$.

Remark 3.5.6. It would also be possible to obtain (3.5.4) essentially from Landweber's classification of invariant prime ideals in MU_* [Lan1, 2.7], or by considering *p-typical* group laws over $\mathbb{Z}_{(p)}$. We have chosen our approach for its more direct emphasis on elementary properties of $[p]$.

We will now conclude the section by applying some of our results thus far to considerations of smoothness and dimension of the algebraic stacks $\mathscr{B}_n$ and $\mathscr{B}_n^{\geq h}$. We shall use freely the language of [LMB], but let us state here explicitly the notion of *relative dimension* of a morphism. We will not (and [LMB] does not) attempt to define the relative dimension of an arbitrary locally finite type morphism of algebraic stacks $f: \mathscr{X} \to \mathscr{Y}$. We can, however, give a satisfactory definition when f is smooth. Indeed, if ξ is a point of $\mathscr{X}$ [LMB, 5.2], then let $\operatorname{Spec} L \to \mathscr{Y}$ be any representative of $f(\xi)$, set $\mathscr{X}_L := \operatorname{Spec} L \times_{\mathscr{Y}} \mathscr{X}$, and let $\widetilde{\xi}$ be any point of $\mathscr{X}_L$ lying over ξ. Then $\mathscr{X}_L$ is a locally Noetherian algebraic stack, and the *relative dimension of f at ξ* is the integer $\dim_\xi f := \dim_{\widetilde{\xi}} \mathscr{X}_L$ [LMB, 11.14]. It is straightforward to verify that the definition is independent of the choices made.

Theorem 3.5.7. *For $n \geq 1$, $\mathscr{B}_n$ is smooth over $\operatorname{Spec}\mathbb{Z}$ of relative dimension -1 at every point. For $h \geq 1$ and $n \geq p^h$, $\mathscr{B}_n^{\geq h}$ is smooth over $\operatorname{Spec}\mathbb{F}_p$ of relative dimension $-h$ at every point.*

Proof. The assertion for $\mathscr{B}_n$ is immediate from the definitions and from $\mathscr{B}_n \approx \mathscr{A}ut(\mathbb{T}_n)\backslash L_n$ (2.3.2), since $L_n \simeq \mathbb{A}_\mathbb{Z}^{n-1}$ and $\mathscr{A}ut(\mathbb{T}_n)$ is an open subscheme of $\mathbb{A}_\mathbb{Z}^n$ (1.6.6). The assertion for $\mathscr{B}_n^{\geq h}$ is similarly immediate from (3.5.1) and (3.5.4). $\square$

3.6. **The stratum of height h n-buds.** In this section we study the *strata* of the height stratification on $\mathscr{B}_n$, or, in other words, the notion of (exact) height for buds. Let $h \geq 1$ and $n \geq p^{h+1}$.

Definition 3.6.1. We denote by $\mathscr{B}_n^h$ the algebraic stack obtained as the open complement of $\mathscr{B}_n^{\geq h+1}$ in $\mathscr{B}_n^{\geq h}$. We call the objects of $\mathscr{B}_n^h$ the n-buds of *height h*, or sometimes of *exact height h*.

Remark 3.6.2. Since formation of infinitesimal neighborhoods is compatible with base change, it's clear from the definitions that height h is stable under truncation (provided we don't truncate below $n = p^{h+1}$). More precisely, an n-bud X has height h $\iff$ $X^{(p^{h+1})}$ has height h.

Let S be a scheme. We can give a more concrete description of the n-buds of height h over S as follows. Since $\mathscr{B}_n^h$ is a stack, height h is a local condition. So, by (1.6.4), the essential case to consider is the n-bud $\mathbb{T}_S^F$ (1.7.5) for some bud law F over $\Gamma(S)$. In this case, we have the pleasing result that the notion of height in (3.6.1) agrees exactly with that for bud laws (3.1.4).

Proposition 3.6.3. $\mathbb{T}_S^F$ *has height h* $\iff$ $[p]_F$ *(3.1.1) is of the form*

$$[p]_F(T) = a_{p^h} T^{p^h} + a_{2p^h} T^{2p^h} + \cdots, \quad a_{p^h} \in \Gamma(S)^\times.$$

Proof. Let $X := \mathbb{T}^F_S$. By (3.1.3) and (3.4.2), X has height $\geq h \iff [p]_F$ is of the asserted form, only with no constraint on a_{p^h}. Now, (3.4.11) and (3.4.13) say that the closed subscheme $Z := \operatorname{Spec} \mathscr{O}_S/a_{p^h}\mathscr{O}_S$ of S is *universal* amongst all S-schemes S' with the property that $S' \times_S X$ is an n-bud of height $\geq h+1$. So X is of height $h \iff \operatorname{Spec} \mathscr{O}_S/a_{p^h}\mathscr{O}_S = \emptyset \iff a_{p^h}$ is a unit. $\square$

Remark 3.6.4. In our discussion of line bundles in §3.3, we could have introduced the notion of principal open substack associated to a section of a line bundle as a sort-of complement to the notion of zero locus of a section of a line bundle. Then (3.6.3) would show, in essence, that $\mathscr{B}^h_n$ can be obtained as the principal open substack in $\mathscr{B}^{\geq h}_n$ associated to v_h (3.4.9). The details are straightforward, but we shall decline to pursue them further.

Remark 3.6.5. In analogy with (3.4.16), (3.6.3) allows us to extend the notion of exact height h to n-buds for $n \geq p^h$, but the added bit of generality again offers no real advantage to us.

Remark 3.6.6. We caution that potential confusion lurks in definitions (3.4.1) and (3.6.1): to say that a bud has "height $\geq h$" is *not* to say that it has "height h' for some $h' \geq h$". For example, if $[p]_F(T) = a_{p^h}T^{p^h} + a_{2p^h}T^{2p^h} + \cdots$ with a_{p^h} a nonzero nonunit, then $\mathbb{T}^F_S$ will have height $\geq h$ but will not have a well-defined height.

Our goal for the remainder of the section is to obtain a characterization of $\mathscr{B}^h_n$. Recall the formal group law $H = H_h$ of (3.1.17) and its n-truncation $H^{(n)}$ (1.7.7). By (3.6.3), the n-bud $\mathbb{T}^{H^{(n)}}_S$ has height h.

Definition 3.6.7. We define $\mathscr{A}ut(H^{(n)})$ to be the presheaf of groups on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{A}ut(H^{(n)})\colon S \longmapsto \operatorname{Aut}_{\Gamma(S)}(H^{(n)}) \cong \operatorname{Aut}_{(n\text{-buds})(S)}\big(\mathbb{T}^{H^{(n)}}_{n,S}\big).$$

Our theorem will assert that $\mathscr{B}^h_n$ is the classifying stack of $\mathscr{A}ut(H^{(n)})$. Now, up to this point, the classifying stacks we've encountered have been essentially independent of the choice of topology; see (2.1.3) and (2.5.6). But our theorem would fail if we only considered $\mathscr{A}ut(H^{(n)})$-torsors for, for example, the Zariski topology. It will be convenient to formulate the theorem in terms of the finite étale topology [SGA3$_\mathrm{I}$, IV 6.3] instead. Quite generally, given a group G over $\operatorname{Spec} \mathbb{F}_p$, we write $B_{\mathrm{f\acute{e}t}}(G)$ for the stack over $(\mathrm{Sch})_{/\mathbb{F}_p}$ of G-torsors for the finite étale topology.

Theorem 3.6.8. $\mathscr{B}^h_n \approx B_{\mathrm{f\acute{e}t}}\big(\mathscr{A}ut(H^{(n)})\big)$.

Proof. By (1.6.4), (1.7.5), (3.1.16), and (3.1.18), $\mathscr{B}^h_n$ is a neutral gerbe over $\operatorname{Spec} \mathbb{F}_p$ for the finite étale topology, with section provided by $\mathbb{T}^{H^{(n)}}_{\mathbb{F}_p}$. So apply [LMB, 3.21]. $\square$

Remark 3.6.9. Since $\mathscr{B}_n$ is a stack for the fpqc topology (2.3.1), so is its locally closed substack $\mathscr{B}^h_n$. Hence we deduce that $B\big(\mathscr{A}ut(H^{(n)})\big)$ is independent of the topology on $(\mathrm{Sch})_{/\mathbb{F}_p}$ between the finite étale and fpqc topologies, inclusive. In particular, every fpqc-torsor for $\mathscr{A}ut(H^{(n)})$ is in fact a finite-étale-torsor.

3.7. **Automorphisms and endomorphisms of buds of height** h. Let $h \geq 1$ and let S be a scheme. For $n \geq p^{h+1}$, by (3.6.8), every n-bud of height h over S is isomorphic finite-étale locally to $\mathbb{T}_S^{H^{(n)}}$, with $H = H_h$ the formal group law of (3.1.17) and $H^{(n)}$ its n-truncation (1.7.7). So we are naturally led to consider closely the group $\mathscr{A}ut(H^{(n)})$ (3.6.7). We shall devote this section to investigating some aspects of its structure. It will be convenient, especially in later sections, to work as much as possible with regard to any $n \geq 1$; but our main results here will require $n \geq p^{h+1}$ (or at least $n \geq p^h$, granting (3.6.5)), so that height h makes sense.

To begin, let $n \geq 1$, and recall the schemes $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ and $(L_n)_{\mathbb{F}_p}$ from §3.5 obtained by reducing $\mathscr{A}ut(\mathbb{T}_n)$ (1.6.5) and L_n (2.2.1), respectively, mod p.

Lemma 3.7.1. $\mathscr{A}ut(H^{(n)})$ *is canonically represented by a closed sub-group scheme of* $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$.

Proof. Roughly, the point is just that $\mathscr{A}ut(H^{(n)})$ naturally sits inside $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ as the stabilizer of $H^{(n)}$. Precisely, $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ acts naturally on $(L_n)_{\mathbb{F}_p}$ by (2.2.2), and we have a Cartesian square

$$
\begin{array}{ccc}
\mathscr{A}ut(H^{(n)}) & \longrightarrow & \mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p} \\
\downarrow & \square & \downarrow \\
\operatorname{Spec}\mathbb{F}_p & \xrightarrow{\quad H^{(n)} \quad} & (L_n)_{\mathbb{F}_p},
\end{array}
$$

where the bottom horizontal arrow is the classifying map determined by $H^{(n)}$ and the right vertical arrow is defined on points by $f \mapsto f \cdot H^{(n)}$. $\square$

By (3.5.7) and (3.6.8), for $n \geq p^{h+1}$, the classifying stack $B\big(\mathscr{A}ut(H^{(n)})\big)$ is an open substack of a stack smooth of relative dimension $-h$ over $\operatorname{Spec}\mathbb{F}_p$. Hence $B\big(\mathscr{A}ut(H^{(n)})\big)$ is itself smooth of relative dimension $-h$ over $\operatorname{Spec}\mathbb{F}_p$. Hence it natural to ask if $\mathscr{A}ut(H^{(n)})$ is smooth of dimension h over $\operatorname{Spec}\mathbb{F}_p$.

We shall answer the question in the affirmative in (3.7.4) below. To prepare, let $n \geq 1$, and recall the $\mathscr{A}_\bullet^{\mathbb{T}_n}$-filtration on $\mathscr{A}ut(\mathbb{T}_n)$ from (1.6.7). Let $(\mathscr{A}_\bullet^{\mathbb{T}_n})_{\mathbb{F}_p}$ denote the filtration on $\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}$ obtained by base change to $\mathbb{F}_p$.

Definition 3.7.2. We define $\mathscr{A}_\bullet^{H^{(n)}}$ to be the intersection filtration on $\mathscr{A}ut(H^{(n)})$:

$$
\mathscr{A}_i^{H^{(n)}} := \mathscr{A}ut(H^{(n)}) \times_{\mathscr{A}ut(\mathbb{T}_n)_{\mathbb{F}_p}} (\mathscr{A}_i^{\mathbb{T}_n})_{\mathbb{F}_p}, \qquad i = 0,\, 1, \ldots,\, n.
$$

Concretely, $\mathscr{A}_0^{H^{(n)}} = \mathscr{A}ut(H^{(n)})$, and $\mathscr{A}_i^{H^{(n)}}$ is given on points by

$$
\mathscr{A}_i^{H^{(n)}}(S) := \left\{ f \in \operatorname{Aut}_{\Gamma(S)}(H^{(n)}) \;\middle|\; \begin{array}{c} f(T) \text{ is of the form} \\ T + a_{i+1}T^{i+1} + a_{i+2}T^{i+2} + \cdots + a_n T^n \end{array} \right\}.
$$

By (1.6.7) and (3.7.1), $\mathscr{A}_0^{H^{(n)}}/\mathscr{A}_1^{H^{(n)}}$ embeds as a closed subscheme of $\mathbb{G}_m = \mathbb{G}_{m,\mathbb{F}_p}$, and $\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}}$ embeds as a closed subscheme of $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{F}_p}$ for $i = 1,\, 2, \ldots,$ $n-1$.

Our main result for the section is following calculation of the successive quotients of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration for $n \geq p^{h+1}$. Let l be the nonnegative integer such that $p^l \leq n < p^{l+1}$.

Theorem 3.7.3. *We have an identification of presheaves*

$$\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}} \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\dots,\ p^{l-h}-1; \\ \mathbb{G}_a, & i = p^{l-h+1}-1,\ p^{l-h+2}-1,\dots,\ p^l-1; \\ 0, & \text{otherwise}; \end{cases}$$

where $\mu_{p^h-1} \subset \mathbb{G}_m$ is the sub-group scheme of (p^h-1)th roots of unity, and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}} \subset \mathbb{G}_a$ is the sub-group scheme of fixed points for the p^hth-power Frobenius operator.

In other words, for any $\mathbb{F}_p$-scheme S, we have

$$\mu_{p^h-1}(S) = \big\{\, a \in \Gamma(S)^\times \mid a^{p^h-1} = 1 \,\big\} \quad \text{and} \quad \mathbb{G}_a^{\mathrm{Fr}_{p^h}}(S) = \big\{\, a \in \Gamma(S) \mid a^{p^h} = a \,\big\}.$$

Hence μ_{p^h-1} and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$ are respresented, respectively, by

$$\operatorname{Spec}\mathbb{F}_p[T]/(T^{p^h-1}-1) \quad \text{and} \quad \operatorname{Spec}\mathbb{F}_p[T]/(T^{p^h}-T).$$

Hence both μ_{p^h-1} and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$ are finite étale groups over $\operatorname{Spec}\mathbb{F}_p$.

Before proceeding to the proof of the theorem, let us first signal an immediate consequence. We continue with $n \geq p^{h+1}$.

Corollary 3.7.4. $\mathscr{A}ut(H^{(n)})$ *is smooth of dimension h over* $\operatorname{Spec}\mathbb{F}_p$.

Proof. By the theorem, $\mathscr{A}ut(H^{(n)})$ is obtained as an iterated extension of smooth groups, so is smooth. Moreover, the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration has precisely h successive quotients of dimension 1, and all other successive quotients of dimension 0. So the dimension assertion follows from [DG, III §3 5.5(a)]. $\qquad\square$

We shall devote the rest of the section to the proof of (3.7.3). One could extract the proof from a careful analysis of some of the statements and arguments in [L, §IV] or in [F, I §3, III §2]. But, for sake of clarity, it seems preferable to give a reasonably self-contained proof here. Our presentation has profited significantly from notes we received from Spallone on a course of Kottwitz.

To prove (3.7.3), it will be somewhat more convenient to translate the problem into one concerning endomorphisms of $H^{(n)}$, as opposed to automorphisms. Let $n \geq 1$.

Definition 3.7.5. We define $\mathscr{E}nd(H^{(n)})$ to be the presheaf of (noncommutative) rings on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{E}nd(H^{(n)}) \colon S \longmapsto \operatorname{End}_{\Gamma(S)}(H^{(n)}) \cong \operatorname{End}_{(n\text{-buds})(S)}\big(\mathbb{T}_{n,S}^{H^{(n)}}\big).$$

Recall that the S-points of $\mathscr{E}nd(H^{(n)})$ are the truncated polynomials $f(T) \in \Gamma_n(S;T)$ that "commute" with $H^{(n)}$ in the sense of (1.1.2). The ring structure on points of $\mathscr{E}nd(H^{(n)})$ is described explicitly in terms of $H^{(n)}$ in (1.1.5).

Definition 3.7.6. For $i = 0,\ 1,\dots,\ n$, we denote by $\mathscr{I}_i^{H^{(n)}}$ the subpresheaf of $\mathscr{E}nd(H^{(n)})$ defined on points by

$$\mathscr{I}_i^{H^{(n)}}(S) := \left\{\, f \in \operatorname{End}_{\Gamma(S)}(H^{(n)}) \,\middle|\, \begin{array}{c} f(T) \text{ is of the form} \\ a_{i+1}T^{i+1}+a_{i+2}T^{i+2}+\cdots+a_nT^n \end{array} \right\}.$$

One verifies immediately that $\mathscr{I}_i^{H^{(n)}}$ is a presheaf of 2-*sided ideals* in $\mathscr{E}nd(H^{(n)})$ for all i, and we have a decreasing filtration

$$\mathscr{E}nd(H^{(n)}) = \mathscr{I}_0^{H^{(n)}} \supset \mathscr{I}_1^{H^{(n)}} \supset \cdots \supset \mathscr{I}_{n-1}^{H^{(n)}} \supset \mathscr{I}_n^{H^{(n)}} = 0.$$

For any $n \geq 1$, we can relate the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration of $\mathscr{E}nd(H^{(n)})$ to the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration of $\mathscr{A}ut(H^{(n)})$ as follows. By (1.1.6), the map on points

$$f \longmapsto \mathrm{id} +_{H^{(n)}} f,$$

where $\mathrm{id}(T) = T$ (1.1.5), defines a morphism of *set*-valued presheaves

$$(*) \qquad\qquad \mathscr{I}_i^{H^{(n)}} \longrightarrow \mathscr{A}_i^{H^{(n)}}, \qquad 1 \leq i \leq n.$$

Lemma 3.7.7. *The arrow* $(*)$ *is an isomorphism of presheaves of sets.*

Proof. The inverse is given by addition with $i_{H^{(n)}}$ (1.1.4). $\qquad\square$

In a moment, we shall exploit the lemma to express the successive quotients of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration in terms of the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration. But we first need another lemma. Quite generally, let R be a possibly noncommutative ring with unit, and let $I \subset R$ be a 2-sided ideal such that $1 + I \subset R^\times$.

Lemma 3.7.8.
  (i) *The natural map* $R^\times/(1+I) \to (R/I)^\times$ *is an isomorphism of groups.*
  (ii) *Let* J *be a 2-sided ideal such that* $I^2 \subset J \subset I$. *Then the map*

$$(**) \qquad\qquad i \longmapsto 1 + i \mod 1 + J$$

  induces an isomorphism of groups $I/J \xrightarrow{\sim} (1+I)/(1+J)$.

Proof. (i) Immediate.

(ii) It plainly suffices to show that $(**)$ defines a group homomorphism $I \to (1+I)/(1+J)$. That is, given i and $i' \in I$, we must find $j \in J$ such that $(1 + i + i')(1 + j) = (1+i)(1+i')$. Take $j := (1 + i + i')^{-1} ii'$. $\qquad\square$

The two previous lemmas yield the following as an immediate consequence.

Lemma 3.7.9. *The natural arrow*

$$\mathscr{A}_0^{H^{(n)}}/\mathscr{A}_1^{H^{(n)}} = \mathscr{A}ut(H^{(n)})/\mathscr{A}_1^{H^{(n)}} \longrightarrow \left(\mathscr{E}nd(H^{(n)})/\mathscr{I}_1^{H^{(n)}}\right)^\times = (\mathscr{I}_0^{H^{(n)}}/\mathscr{I}_1^{H^{(n)}})^\times$$

is an isomorphism of presheaves of abelian groups. For $1 \leq i \leq n-1$, *the arrow* $(*)$ *induces an isomorphism of presheaves of abelian groups*

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \xrightarrow{\sim} \mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}}. \qquad\qquad\square$$

This last lemma reduces (3.7.3) to the calculation of the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration. To prepare, it will be convenient to formulate a few general lemmas on homomorphisms between bud laws. We continue with $n \geq 1$.

Let A be a ring, and fix n-bud laws F and G over A. Let $f \in A[T]/(T)^{n+1}$ have 0 constant term. We define $\partial f \in A[T_1, T_2]/(T_1, T_2)^{n+1}$ to measure the failure of f to be a homomorphism $F \to G$:

$$(\dagger) \qquad\qquad (\partial f)(T_1, T_2) := f\big(F(T_1, T_2)\big) - G\big(f(T_1), f(T_2)\big).$$

In keeping with our notation for truncated bud and group laws (1.7.7), we write $f^{(m)}$ for the image of f in $A[T]/(T)^{m+1}$, and $(\partial f)^{(m)}$ for the image of ∂f in $A[T_1, T_2]/(T_1, T_2)^{m+1}$, $m \leq n$.

As an easy first lemma, we shall consider the effect of perturbations to f on ∂f. Let $a \in A$, and put $g(T) := f(T) + aT^m$. Recall the polynomial B_m of (3.1.11).

Lemma 3.7.10. $(\partial g)^{(m)} = (\partial f)^{(m)} + aB_m$.

Proof. Without loss of generality, we may assume $m = n$. Then we just compute

$$g\big(F(T_1, T_2)\big) = f\big(F(T_1, T_2)\big) + aF(T_1, T_2)^n = f\big(F(T_1, T_2)\big) + a(T_1 + T_2)^n$$

and

$$G\big(g(T_1), g(T_2)\big) = G\big(f(T_1) + aT^n, f(T_2) + aT^n\big) = G\big(f(T_1), f(T_2)\big) + aT_1^n + aT_2^n$$

and subtract. $\qquad\square$

For the next two lemmas, assume $m \geq 2$, and suppose that $f^{(m-1)}$ is a homomorphism $F^{(m-1)} \to G^{(m-1)}$, so that $(\partial f)^{(m-1)} = 0$. Recall the polynomial C_m of (3.1.11).

Lemma 3.7.11. *There exists a unique $a \in A$ such that $(\partial f)^{(m)} = aC_m$.*

Proof. We shall show that $(\partial f)^{(m)}$ is a homogenous SPC (3.1.9) of degree m and appeal to (3.1.12). Without loss of generality, we may assume $m = n$. Since F and G are commutative, it is clear that ∂f is symmetric. Since $f^{(n-1)}$ is a homomorphism, it is clear that ∂f is homogenous of degree n. So we just have to show that ∂f satisfies the cocycle condition (3.1.9C). We shall do so by exploiting the associativity of F and G.

Replacing T_2 by $F(T_2, T_3)$ in (†), we obtain an equality of elements in the ring $A[T_1, T_2, T_3]/(T_1, T_2, T_3)^{n+1}$,

$$(\partial f)\big(T_1, F(T_2, T_3)\big) = f\Big(F\big(T_1, F(T_2, T_3)\big)\Big) - G\Big(f(T_1), f\big(F(T_2, T_3)\big)\Big).$$

Replacing $f\big(F(T_2, T_3)\big)$ with $G\big(f(T_2), f(T_3)\big) + (\partial f)(T_2, T_3)$ in the display, we obtain

$$(\partial f)\big(T_1, F(T_2, T_3)\big) = f\Big(F\big(T_1, F(T_2, T_3)\big)\Big)$$
$$- G\Big(f(T_1), G\big(f(T_2), f(T_3)\big) + (\partial f)(T_2, T_3)\Big).$$

Since ∂f is homogenous of degree n, the left-hand side of this last display is

$$(\partial f)(T_1, T_2 + T_3),$$

and the right-hand side is

$$f\Big(F\big(T_1, F(T_2, T_3)\big)\Big) - G\Big(f(T_1), G\big(f(T_2), f(T_3)\big)\Big) - (\partial f)(T_2, T_3).$$

Analogously, replacing T_1 by $F(T_1, T_2)$ and T_2 by T_3 in (†), one obtains a second equality in $A[T_1, T_2, T_3]/(T_1, T_2, T_3)^{n+1}$,

$$(\partial f)(T_1 + T_2, T_3) = f\Big(F\big(F(T_1, T_2), T_3\big)\Big)$$
$$- G\Big(G\big(f(T_1), f(T_2)\big), f(T_3)\Big) - (\partial f)(T_1, T_2).$$

Now subtract equations and use the associativity of F and G. $\qquad\square$

We continue with a as in the previous lemma. Recall $\lambda(m)$ from (3.1.11).

Lemma 3.7.12. *For any $k \geq 1$,*

$$(f^{(m)} \circ [k]_{F^{(m)}})(T) = ([k]_{G^{(m)}} \circ f^{(m)})(T) + \frac{k^m - k}{\lambda(m)} \cdot aT^m.$$

In particular, for $k = p$, m of the form p^j, and A of characteristic p, we have

$$(f^{(p^j)} \circ [p]_{F^{(p^j)}})(T) = ([p]_{G^{(p^j)}} \circ f^{(p^j)})(T) - aT^m.$$

Proof. The proof is entirely similar to that given in [L, Lemme 6] or in [F, III §1 Lemma 4], but for convenience, we'll write out the details. Without loss of generality, we may assume $m = n$.

We induct on k. The assertion is clear for $k = 1$. So assume the assertion holds for k. We have

$$(f \circ [k+1]_F)(T) = f\Big(F\big(T, [k]_F(T)\big)\Big)$$
$$= G\big(f(T), (f \circ [k]_F)(T)\big) + (\partial f)\big(T, [k]_F(T)\big) \qquad (\dagger).$$

Let us examine separately the two terms appearing in the last expression. We have

$$G\big(f(T), (f \circ [k]_F)(T)\big) = G\Big(f(T), ([k]_G \circ f)(T) + \tfrac{k^n-k}{\lambda(n)}aT^n\Big) \qquad \text{(induction)}$$
$$= G\big(f(T), ([k]_G \circ f)(T)\big) + \frac{k^n - k}{\lambda(n)}aT^n$$
$$= ([k+1]_G \circ f)(T) + \frac{k^n - k}{\lambda(n)}aT^n.$$

Moreover,

$$(\partial f)\big(T, [k]_F(T)\big) = aC_n\big(T, [k]_F(T)\big) \qquad (3.7.11)$$
$$= aC_n(T, kT) \qquad (3.1.2)$$
$$= a\frac{(k+1)^n - k^n - 1}{\lambda(n)}T^n.$$

The lemma follows at once. $\qquad \qquad \square$

We are now ready to compute the $\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}}$'s. We denote by $\mathbb{O} = \mathbb{O}_{\mathbb{F}_p}$ the tautological ring scheme structure on $\mathbb{A}^1_{\mathbb{F}_p}$, and by $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ the sub-ring scheme of $\mathbb{O}$ of fixed points for the p^hth-power Frobenius operator:

$$\mathbb{O}^{\mathrm{Fr}_{p^h}}(S) = \big\{ a \in \Gamma(S) \mid a^{p^h} = a \big\}.$$

Quite as in (1.6.7), one verifies at once that the map on points

$$(\ddagger) \qquad\qquad a_{i+1}T^{i+1} + \cdots + a_n T^n \longmapsto a_{i+1}$$

specifies a monomorphism of presheaves of rings

$$\mathscr{I}_0^{H^{(n)}}/\mathscr{I}_1^{H^{(n)}} \hookrightarrow \mathbb{O}, \qquad i = 0,$$

and a monomorphism of presheaves of abelian groups

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \hookrightarrow \mathbb{G}_a, \qquad 1 \leq i \leq n - 1.$$

Assume $n \geq p^{h+1}$, and again let l be the nonnegative integer such that $p^l \leq n < p^{l+1}$.

Theorem 3.7.13. *For $0 \le i \le n-1$, ($\ddagger$) induces an identification of presheaves*

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \cong \begin{cases} \mathbb{O}^{\mathrm{Fr}_{p^h}}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ldots,\ p^{l-h}-1; \\ \mathbb{G}_a, & i = p^{l-h+1}-1,\ p^{l-h+2}-1,\ldots,\ p^l-1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let A be a ring of characteristic p, and let $I_i := \mathscr{I}_i^{H^{(n)}}(A)$, $0 \le i \le n$. For $i \ne 0,\ p-1,\ p^2-1,\ldots,\ p^l-1$, we have $I_i/I_{i+1} = 0$ by (3.1.3). So we are left to compute the quotients for i of the form $p^j - 1$.

As a first step, let $f(T) = a_1 T + \cdots a_n T^n$ be any endomorphism of $H^{(n)}$ over A. Let $A^{\mathrm{Fr}_{p^h}} := \mathbb{O}^{\mathrm{Fr}_{p^h}}(A)$. Since f must commute with $[p]_{H^{(n)}}(T) = T^{p^h}$, we deduce $a_i \in A^{\mathrm{Fr}_{p^h}}$ for $ip^h \le n$. In particular, the map ($\ddagger$) carries I_i/I_{i+1} into $A^{\mathrm{Fr}_{p^h}}$ for $i = 0,\ p-1,\ldots,\ p^{l-h}-1$, as asserted. So we are reduced to showing the following: given $a_{p^j}T^{p^j} \in A[T]/(T)^{n+1}$, with $a_{p^j} \in A^{\mathrm{Fr}_{p^h}}$ in case $j \le l-h$ and no constraint on a_{p^j} in case $j > l-h$, we can add terms of degree $> p^j$ to obtain an endomorphism of $H^{(n)}$.

We shall proceed by induction on the degree of the term to be added. We work with ∂ ($\dagger$) computed with regard to $F = G = H^{(n)}$. To begin, let $f(T) = a_{p^j}T^{p^j}$. Then

$$(\partial f)^{(p^j)}(T_1, T_2) = f\big(H^{(p^j)}(T_1, T_2)\big) - H^{(p^j)}\big(f(T_1), f(T_2)\big)$$
$$= a_{p^j}(T_1 + T_2)^{p^j} - a_{p^j}T_1^{p^j} - a_{p^j}T_2^{p^j}$$
$$= 0.$$

Hence f defines an endomorphism of $H^{(p^j)}$. We must now show that if $g(T) = a_{p^j}T^{p^j} + \cdots + a_{m-1}T^{m-1}$ defines an endomorphism of $H^{(m-1)}$, $p^j + 1 \le m \le n$, then we can always add a term of degree m to g to obtain an endomorphism of $H^{(m)}$.

Case 1: m is not a power of p. By (3.7.11), $(\partial g)^{(m)} = aC_m$ for a unique $a \in A$. Now, since A is of characteristic p and m is not a power of p, C_m is a unit multiple of B_m over A. Hence by (3.7.10) we can find (a unique) $a_m \in A$ such that $g(T) + a_m T^m$ defines an endomorphism of $H^{(m)}$.

Note that if $a_{p^j},\ldots,a_{m-1} \in A^{\mathrm{Fr}_{p^h}}$, then $a_m \in A^{\mathrm{Fr}_{p^h}}$ too, since H is defined over $\mathbb{F}_p$, $A^{\mathrm{Fr}_{p^h}}$ is a subring of A, and a_m is uniquely determined.

Case 2: m is a power of p. In this case $B_m = 0$ over A, so the method of case 1 breaks down. But we claim that, in fact, g is already an endomorphism of $H^{(m)}$. By (3.7.11) and (3.7.12), it suffices to show that $g \circ [p]_{H^{(m)}} = [p]_{H^{(m)}} \circ g$ in $A[T]/(T)^{m+1}$. We shall show that the stronger statement $g \circ [p]_{H^{(n)}} = [p]_{H^{(n)}} \circ g$ in $A[T]/(T)^{n+1}$ holds. We consider separately the subcases $j \le l-h$ and $j > l-h$. If $j \le l-h$, then recall $a_{p^j} \in A^{\mathrm{Fr}_{p^h}}$. Hence by induction all the coefficients of g lie in $A^{\mathrm{Fr}_{p^h}}$. Hence g commutes with $[p]_{H^{(n)}}$. If $j > l-h$, then the argument is even easier: $g \circ [p]_{H^{(n)}}$ and $[p]_{H^{(n)}} \circ g$ both only involve terms of degrees $\ge p^{j+h}$. So both are 0 in $A[T]/(T)^{n+1}$ by definition of l. $\qquad\square$

At last we obtain the proof of (3.7.3).

Proof of (3.7.3). Clear from (3.7.9) and (3.7.13), noting for the $i = 0$ case that μ_{p^h-1} sits naturally inside $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ as the subfunctor of units. $\qquad\square$

Remark 3.7.14. One verifies immediately that the maps

$$\mathscr{A}_i^{H^{(n)}}/\mathscr{A}_{i+1}^{H^{(n)}} \longrightarrow \begin{cases} \mathbb{G}_m, & i = 0; \\ \mathbb{G}_a, & 1 \leq i \leq n-1 \end{cases}$$

induced by (1.6.7) and the maps

$$\mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow \begin{cases} \mathbb{O}, & i = 0; \\ \mathbb{G}_a, & 1 \leq i \leq n-1 \end{cases}$$

of the previous theorem are compatible with the identifications of (3.7.9).

4. The height stratification: formal Lie groups

We continue working with respect to fixed prime p.

4.1. **The height stratification on the stack of formal Lie groups.** In this section we introduce the height stratification on the stack of formal Lie groups, quite in analogy with the height stratification on $\mathscr{B}_n$, $n \geq 1$.

Let $h \geq 0$. We denote by $\mathscr{FLG}^{\geq h}$ the full sub-fibered category of $\mathscr{FLG}$ rendering the diagram

$$(*) \qquad \begin{array}{ccc} \mathscr{FLG}^{\geq h} & \longrightarrow & \mathscr{FLG} \\ \downarrow & & \downarrow \\ \mathscr{B}_{p^h}^{\geq h} & \longrightarrow & \mathscr{B}_{p^h} \end{array}$$

Cartesian; here, as usual, the right vertical arrow denotes truncation. Abusing notation, we denote again by $\mathscr{L}$ the pullback to $\mathscr{FLG}$ of the line bundle $\mathscr{L}$ on $\mathscr{B}_1$ (3.4.6). Similarly, we abusively denote by $\mathscr{L}_h$ the restriction of $\mathscr{L}$ to $\mathscr{FLG}^{\geq h}$; then $\mathscr{L}_h$ is canonically isomorphic to the pullback to $\mathscr{FLG}^{\geq h}$ of the line bundle $\mathscr{L}_h$ on $\mathscr{B}_{p^h}^{\geq h}$ (3.4.6). We abusively denote again by v_h the section $\mathscr{O}_{\mathscr{FLG}^{\geq h}} \to \mathscr{L}_h^{\otimes p^h - 1}$ over $\mathscr{FLG}^{\geq h}$ obtained by pulling back the section $v_h \colon \mathscr{O}_{\mathscr{B}_{p^h}^{\geq h}} \to \mathscr{L}_h^{\otimes p^h - 1}$ from $\mathscr{B}_{p^h}^{\geq h}$ (3.4.9).

The fibered category $\mathscr{FLG}^{\geq h}$ and the various sections v_i are related in the following simple way. Let X be a formal Lie group over the base scheme S.

Proposition 4.1.1. *The following are equivalent.*
- (i) *X is an object in $\mathscr{FLG}^{\geq h}$.*
- (ii) *The p^h-bud $X^{(p^h)}$ has height $\geq h$.*
- (iii) *For any $n \geq p^h$, the n-bud $X^{(n)}$ has height $\geq h$.*
- (iv) *X is an object in each of the successive zero loci (3.3.3) $V(v_0)$, $V(v_1), \ldots,$ $V(v_{h-1})$.*

Proof. (3.4.4) and (3.4.13). □

Definition 4.1.2. X has *height* $\geq h$ it it satisfies the equivalent conditions of (4.1.1).

Example 4.1.3. Quite as for buds (3.4.2), given a formal group law F over $\Gamma(S)$, the formal Lie group $\widehat{\mathbb{A}}_S^F$ (1.3.3) has height $\geq h \iff [p]_F \in T^{p^h} \cdot \Gamma(S)[[T]]$. So our terminology again comports with (3.1.4).

Remark 4.1.4. Many of the above definitions are independent of fixed choices we've made. For example, the proposition says that we could have just as well defined $\mathscr{FLG}^{\geq h}$ by replacing the diagram $(*)$ with one in which p^h is everywhere replaced by any $n \geq p^h$. Up to canonical isomorphism, (3.4.8) says we could have defined $\mathscr{L}$ as the pullback to $\mathscr{FLG}$ of the line bundle $\mathscr{L}$ on $\mathscr{B}_n$, for any $n \geq 1$; and similarly for $\mathscr{L}_h$, for any $n \geq p^h$. Analogously, (3.4.12) says that we could have defined v_h as the pullback of the section v_h over $\mathscr{B}_n^{\geq h}$, for any $n \geq p^h$.

Remark 4.1.5. Just as for buds, $\mathscr{FLG}^{\geq 0} = \mathscr{FLG}$, and $\mathscr{FLG}^{\geq 1}$ is the stack of formal Lie groups over $\mathbb{F}_p$-schemes.

Proposition 4.1.6. $\mathscr{FLG}^{\geq h}$ *is a stack for the fpqc topology, and the inclusion functor* $\mathscr{FLG}^{\geq h} \to \mathscr{FLG}$ *is a closed immersion.*

Proof. The diagram $(*)$ is Cartesian. So the first assertions follows because $\mathscr{B}_{p^h}$ (2.3.1), $\mathscr{FLG}$ (2.6.6), and $\mathscr{B}_{p^h}^{\geq h}$ (3.4.14) are fpqc stacks. And the second assertion follows because $\mathscr{B}_{p^h}^{\geq h} \to \mathscr{B}_{p^h}$ is a closed immersion (3.4.14). $\square$

Remark 4.1.7. As for buds, we obtain a decreasing filtration of closed substacks $\mathscr{FLG} = \mathscr{FLG}^{\geq 0} \supsetneq \mathscr{FLG}^{\geq 1} \supsetneq \mathscr{FLG}^{\geq 2} \supsetneq \cdots$. By contrast with the bud case (3.4.4), the filtration for $\mathscr{FLG}$ is of infinite length.

4.2. **The stack of height $\geq h$ formal Lie groups.** In this section we collect some characterizations of $\mathscr{FLG}^{\geq h}$ analogous to previous results on $\mathscr{B}_n^{\geq h}$ and $\mathscr{FLG}$.

Let us first consider an analog to the description of $\mathscr{B}_n^{\geq h}$ in (3.5.1). Let U be a universal (for $\mathbb{Z}_{(p)}$-algebras) formal group law over $\mathbb{Z}_{(p)}[t_1, t_2, \dots]$ as in (3.5.5). Recall that, in the notation of (3.5.5), the reduction of U over the ring

$$B_h := \mathbb{Z}_{(p)}[t_1, t_2, \dots]/(p, a_1, \dots, a_{h-1}) \cong \mathbb{F}_p[t_1, t_2, \dots]/(\overline{a}_1, \dots, \overline{a}_{h-1})$$

is a universal group law of height $\geq h$, and B_h is a polynomial ring over $\mathbb{F}_p$ on the images of the t_i for $i \neq p^1 - 1, p^2 - 1, \dots, p^{h-1} - 1$. Let $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} := \mathscr{A}ut(\widehat{\mathbb{A}}) \otimes \mathbb{F}_p$, with $\mathscr{A}ut(\widehat{\mathbb{A}})$ as in (2.5.3). Just as in (3.5.1), we deduce the following.

Theorem 4.2.1. $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$ *acts naturally on* $\operatorname{Spec} B_h$, *and we have* $\mathscr{FLG}^{\geq h} \approx \mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} \backslash \operatorname{Spec} B_h$. $\square$

In analogy with (2.6.8), let us next consider the relation between the stacks $\mathscr{FLG}^{\geq h}$ and $\mathscr{B}_n^{\geq h}$, $n \geq p^h$. By (3.4.4), we may form the limit $\varprojlim_{n \geq p^h} \mathscr{B}_n^{\geq h}$ of the $\mathscr{B}_n^{\geq h}$'s with respect to the truncation functors. By (4.1.1), truncation determines an arrow

$$(*) \qquad\qquad \mathscr{FLG}^{\geq h} \longrightarrow \varprojlim_{n \geq p^h} \mathscr{B}_n^{\geq h}.$$

As always, we emphasize that the limit is taken in the sense of bicategories; see the appendix.

Theorem 4.2.2. *The arrow $(*)$ is an equivalence of stacks.*

Proof. Combine (2.6.8), (4.1.1), and (A.9.2). $\square$

4.3. **The stratum of height h formal Lie groups I.** In this section, in analogy with §3.6, we begin to study the *strata* of the height stratification on $\mathscr{FLG}$, or, in other words, the notion of (exact) height for formal Lie groups. Let X be a formal Lie group over the base scheme S.

Proposition 4.3.1. *The following are equivalent.*

   (i) *The p^{h+1}-bud $X^{(p^{h+1})}$ has height h.*
   (ii) *For any $n \geq p^{h+1}$, the n-bud $X^{(n)}$ has height h.*
   (iii) *X is an object in the open complement of $\mathscr{FLG}^{\geq h+1}$ in $\mathscr{FLG}^{\geq h}$.*

Proof. (3.6.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 4.3.2. X has *height h*, or *exact height h*, if it satisfies the equivalent conditions of (4.3.1). We denote by $\mathscr{FLG}^h$ the substack of $\mathscr{FLG}$ of formal Lie groups of height h.

Example 4.3.3. Quite as for buds (3.6.3), if $X = \widehat{\mathbb{A}}_S^F$ for the formal group law F over $\Gamma(S)$ (1.3.3), then the notion of height h for X recovers precisely that for F (3.1.4).

Remark 4.3.4. The caution of (3.6.6) still applies: to say that a formal Lie group has "height $\geq h$" is *not* to say that it has "height h' for some $h' \geq h$".

Remark 4.3.5 (Relation to p-Barsotti-Tate groups). Our notion of height for formal Lie groups is related to, but *not* strictly compatible with, the notion of height for p-Barsotti-Tate, or p-divisible, groups. In rough form, the difference is that (exact) height for formal Lie groups is a *locally closed* condition, whereas height for Barsotti-Tate groups is a *fiberwise* condition. For example, if X is a formal Lie group of height h in the sense of (4.3.2), then X is a Barsotti-Tate group of height h in the sense of Barsotti-Tate groups. But the converse can easily fail. For example, $\widehat{\mathbb{G}}_m$ is a Barsotti-Tate group of height 1 over any base scheme on which p is locally nilpotent. But $\widehat{\mathbb{G}}_m$ has height 1 as a formal Lie group exactly when p is honestly 0.

Similar examples exist for any height $h > 1$. Let A be a ring and $I \subset A$ a nonzero nilideal such that $B := A/I$ is of characteristic p. Let F be a formal group law of height h over B. As A is necessarily a $\mathbb{Z}_{(p)}$-algebra, we may apply (3.5.5) to lift F to a group law $\widetilde{F}$ over A such that the coefficients in $[p]_{\widetilde{F}}(T)$ of T^p, T^{p^2}, ..., $T^{p^{h-1}}$ are any elements of I that we like. In particular, we can ensure that $\widetilde{F}$ is not of height $\geq h$. But it is easy to verify that $\widehat{\mathbb{A}}_A^{\widetilde{F}}$ (1.3.3) is a Barsotti-Tate group of height h.

Remark 4.3.6 (Relation to p-typical formal group laws). Let us digress for a moment to briefly discuss BP-theory and p-typical formal group laws. We refer to [R1] for general background, especially to [R1, App. 2] for the relevant group law theory. Recall that BP_* and the ring $W := BP_*[t_0, t_0^{-1}, t_1, t_2, \dots]$ admit a natural Hopf algebroid structure such that the associated internal groupoid in the category of affine $\mathbb{Z}_{(p)}$-schemes

$$(*) \qquad\qquad\qquad \operatorname{Spec} W \rightrightarrows \operatorname{Spec} BP_*$$

represents p-typical formal group laws and the isomorphisms between them. In particular, letting $\mathscr{X}$ denote the stackification of $(*)$, there is a natural morphism $f \colon \mathscr{X} \to \mathscr{FLG} \otimes \mathbb{Z}_{(p)}$, and one verifies just as in [N, 34(2)] that f is an equivalence.

Hence the height stratification on $\mathscr{FLG}$ induces a stratification on $\mathscr{X}$, or in other words, a stratification on $\operatorname{Spec} BP_*$ by *invariant* closed subschemes.

Now, recall that $BP_* \simeq \mathbb{Z}_{(p)}[u_1, u_2, \dots]$, where for convenience we take the u_i's to be the Araki generators and set $u_0 := p$. Recall also Landweber's ideals $I_0 := 0$ and $I_h := (u_0, u_1, \dots, u_{h-1})$, $h > 0$, in BP_*. Then for all $h \geq 0$, *the closed substack* $\mathscr{FLG}^{\geq h} \otimes \mathbb{Z}_{(p)}$ *in* $\mathscr{FLG} \otimes \mathbb{Z}_{(p)} \approx \mathscr{X}$ *corresponds to the ideal* $I_h \subset BP_*$; one may deduce this essentially from Landweber's classification of invariant prime ideals in BP_* [Lan1, 2.7; Lan2, 6.2], or in a more direct fashion from the formula [R1, A2.2.4] (this formula is the only point where our particular choice of the Araki generators enters). In particular, our notion of (exact) height agrees with Pribble's [P, 4.5]. The identification of the height stratification and the I_h-stratification on $\mathscr{X}$ is also noted in [N, §6 pp. 25–26]; one verifies immediately that Naumann's definition of the height stratification agrees with ours.

This said, let us note that our notion of height is not completely compatible with the notion of height for BP_*-algebras in [HS, 4.1]. Namely, given a BP_*-algebra A, consider the composite

$$\operatorname{Spec} A \longrightarrow \operatorname{Spec} BP_* \longrightarrow \mathscr{FLG}.$$

From the point of view of this paper, it would be reasonable to say that A is a BP_*-algebra of height h if the displayed composite factors through $\mathscr{FLG}^h$. But, as noted in [N, 24], A has height h in the sense of [HS] if it satisfies the strictly weaker condition that h is the smallest nonnegative integer for which the composite factors through the open substack $\mathscr{FLG} - \mathscr{FLG}^{\geq h+1}$ of $\mathscr{FLG}$ ([HS] defines A to have height ∞ if the composite fails to factor through $\mathscr{FLG} - \mathscr{FLG}^{\geq h+1}$ for any h).

We shall next formulate a characterization of $\mathscr{FLG}^h$ analogous to (3.6.8). Recall the formal group law $H = H_h$ of (3.1.17).

Definition 4.3.7. We define $\mathscr{A}ut(H)$ to be the presheaf of groups on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{A}ut(H) \colon S \longmapsto \operatorname{Aut}_{\Gamma(S)}(H) \cong \operatorname{Aut}_{(\mathrm{FLG})(S)}\big(\widehat{\mathbb{A}}_S^H\big).$$

Whereas in (3.6.8) we were led to consider torsors for the finite étale topology, we shall now need to consider $\mathscr{A}ut(H)$-torsors for the fpqc topology. Given a group G over $\operatorname{Spec} \mathbb{F}_p$, we write $B_{\mathrm{fpqc}}(G)$ for the stack over $(\mathrm{Sch})_{/\mathbb{F}_p}$ of G-torsors for the fpqc topology.

Theorem 4.3.8. $\mathscr{FLG}^h \approx B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)\big)$.

Proof. Essentially identical to the proof of (3.6.8). $\qquad\square$

Remark 4.3.9. The statement of the theorem is not entirely sharp: by (3.1.18), it would suffice to replace the fpqc topology by the topology on $(\mathrm{Sch})_{/\mathbb{F}_p}$ generated by the Zariski topology and all maps $\operatorname{Spec} B \to \operatorname{Spec} A$ between affine schemes obtained as a limit of surjective finite étale maps $\cdots \to \operatorname{Spec} B_2 \to \operatorname{Spec} B_1 \to \operatorname{Spec} A$.

We shall study the group $\mathscr{A}ut(H)$ and its relation to the groups $\mathscr{A}ut(H^{(n)})$ for varying $n \geq p^{h+1}$ in the next section.

We shall conclude this section by formulating another characterization of the stack $\mathscr{FLG}^h$, this time the obvious analog of (4.2.2). By (3.6.2), we may form the

limit $\varprojlim_{n \geq p^{h+1}} \mathscr{B}_n^h$ of the $\mathscr{B}_n^h$'s with respect to the truncation functors. By (4.3.1), truncation determines an arrow

$$(**) \qquad\qquad \mathscr{FLG}^h \longrightarrow \varprojlim_{n \geq p^{h+1}} \mathscr{B}_n^h.$$

As always, we emphasize that the limit is taken in the sense of bicategories; see the appendix. As in (4.2.2), only replacing the reference to (4.1.1) with (4.3.1), we obtain the following.

Theorem 4.3.10. *The arrow* $(**)$ *is an equivalence of stacks.* $\square$

4.4. **Automorphisms and endomorphisms of formal Lie groups of height** h. Let $h \geq 1$. Our result $\mathscr{FLG}^h \approx B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)\big)$ (4.3.8), with $\mathscr{FLG}^h$ the stratum in $\mathscr{FLG}$ of formal Lie groups of height h, leads us to consider closely the $\mathbb{F}_p$-group $\mathscr{A}ut(H)$ (4.3.7). We shall devote this section to investigating some aspects of its structure and of its relation to the groups $\mathscr{A}ut(H^{(n)})$ (3.6.7), $n \geq 1$. We shall ultimately apply our final result of this section, (4.4.11), to obtain another characterization of $\mathscr{FLG}^h$ in §4.6.

Let us begin with the analog of (3.7.1) for $\mathscr{A}ut(H)$. Recall the $\mathbb{Z}$-group $\mathscr{A}ut(\widehat{\mathbb{A}})$ (2.5.3), and let $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p} := \mathscr{A}ut(\widehat{\mathbb{A}}) \otimes \mathbb{F}_p$. Quite as in (3.7.1), we obtain the following.

Lemma 4.4.1. $\mathscr{A}ut(H)$ *is canonically represented by a closed sub-group scheme of* $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$. $\square$

Quite as in §3.7, although we will ultimately be interested in automorphisms of H, we shall accord the endomorphisms of H a more fundamental role.

Definition 4.4.2. We define $\mathscr{E}nd(H)$ to be the presheaf of (noncommutative) rings on $(\mathrm{Sch})_{/\mathbb{F}_p}$

$$\mathscr{E}nd(H) \colon S \longmapsto \mathrm{End}_{\Gamma(S)}(H) \cong \mathrm{End}_{(\mathrm{FLG})(S)}\big(\widehat{\mathbb{A}}_S^H\big).$$

The ring structure on points of $\mathscr{E}nd(H)$ is described explicitly in terms of H in (1.1.5).

The $\mathscr{I}_\bullet^{H^{(n)}}$-filtration on $\mathscr{E}nd(H^{(n)})$ (3.7.6) admits a natural analog for $\mathscr{E}nd(H)$, as follows.

Definition 4.4.3. For $i = 0, 1, 2, \ldots$, we denote by $\mathscr{I}_i^H$ the subpresheaf of $\mathscr{E}nd(H)$ defined on points by

$$\mathscr{I}_i^H(S) := \left\{ f \in \mathrm{End}_{\Gamma(S)}(H) \;\middle|\; \begin{array}{c} f(T) \text{ is of the form} \\ a_{i+1}T^{i+1} + (\text{higher order terms}) \end{array} \right\}.$$

Quite as for $\mathscr{I}_i^{H^{(n)}}$, one verifies immediately that $\mathscr{I}_i^H$ is a presheaf of 2-*sided ideals* in $\mathscr{E}nd(H)$ for all i, and we have a decreasing filtration

$$\mathscr{E}nd(H) = \mathscr{I}_0^H \supset \mathscr{I}_1^H \supset \mathscr{I}_2^H \supset \cdots,$$

this time of infinite length.

We now wish to introduce the analog for $\mathscr{A}ut(H)$ of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration on $\mathscr{A}ut(H^{(n)})$ (3.7.2). We could do so by mimicking the definition of the $\mathscr{A}_\bullet^{H^{(n)}}$-filtration in the obvious way: there is a natural filtration on $\mathscr{A}ut(\widehat{\mathbb{A}})$ in plain analogy with (1.6.7), hence an induced filtration on $\mathscr{A}ut(\widehat{\mathbb{A}})_{\mathbb{F}_p}$, hence an intersection filtration on $\mathscr{A}ut(H)$. Instead, we will just use directly the $\mathscr{I}_\bullet^H$-filtration on $\mathscr{E}nd(H)$.

Definition 4.4.4. We define $\mathscr{A}_i^H$ to be the subpresheaf of $\mathscr{E}nd(H)$

$$\mathscr{A}_i^H := \begin{cases} \mathscr{A}ut(H), & i = 0 \\ T +_H \mathscr{I}_i^H, & i = 1,\, 2,\dots. \end{cases}$$

Concretely, reasoning as in (3.7.7), $\mathscr{A}_i^H$ is given on points by

$$\mathscr{A}_i^H(S) := \left\{ f \in \mathrm{Aut}_{\Gamma(S)}(H) \,\middle|\, \begin{array}{c} f(T) \text{ is of the form} \\ T + a_{i+1}T^{i+1} + (\text{higher order terms}) \end{array} \right\}.$$

It is immediate that $\mathscr{A}_i^H$ is a *normal subgroup* in $\mathscr{A}ut(H)$ for all i, and we have a decreasing filtration

$$\mathscr{A}ut(H) = \mathscr{A}_0^H \supset \mathscr{A}_1^H \supset \mathscr{A}_2^H \supset \cdots.$$

Let us now turn to the relation between $\mathscr{E}nd(H)$ and the $\mathscr{E}nd(H^{(n)})$'s, and between $\mathscr{A}ut(H)$ and the $\mathscr{A}ut(H^{(n)})$'s. For any $m \geq n \geq 1$, truncation of H induces a commutative diagram of presheaves of rings

$$(*)$$

$$\begin{array}{ccc} & \mathscr{E}nd(H) & \\ & \swarrow \qquad \searrow & \\ \mathscr{E}nd(H^{(m)}) & \longrightarrow & \mathscr{E}nd(H^{(n)}). \end{array}$$

Proposition 4.4.5. *For all $i \geq 0$, the diagram $(*)$ induces*

(i) $\mathscr{I}_i^H \xrightarrow{\sim} \varprojlim_{n \geq 1} \mathscr{I}_i^{H^{(n)}}$, *where we take $\mathscr{I}_i^{H^{(n)}} := 0$ for $i \geq n$; and*

(ii) $\mathscr{A}_i^H \xrightarrow{\sim} \varprojlim_{n \geq 1} \mathscr{A}_i^{H^{(n)}}$, *where we take $\mathscr{A}_i^{H^{(n)}} := 1$ for $i \geq n$.*

Moreover,

(iii) $\mathscr{I}_i^H \xrightarrow{\sim} \varprojlim_{n \geq i} \mathscr{I}_i^H/\mathscr{I}_n^H$ *and*

(iv) $\mathscr{A}_i^H \xrightarrow{\sim} \varprojlim_{n \geq i} \mathscr{A}_i^H/\mathscr{A}_n^H$.

In particular, $\mathscr{E}nd(H)$ (resp. $\mathscr{A}ut(H)$) is complete and separated with respect to the $\mathscr{I}_\bullet^H$- (resp. $\mathscr{A}_\bullet^H$-) topology.

Proof. Before anything else, it is clear from the definitions that truncation carries $\mathscr{I}_i^H$ and $\mathscr{I}_i^{H^{(m)}}$ into $\mathscr{I}_i^{H^{(n)}}$, $m \geq n$, so that the limit and arrow in (i) are well-defined; and analogously for (ii).

(i) The case $i = 0$ is clear because $\widehat{\mathbb{A}}^H$ (1.3.3) is ind-infinitesimal: precisely, use (2.6.8), (A.6.2), and (A.4.4). The case $i > 0$ is then clear because, for all $n \geq i$, the inverse image of $\mathscr{I}_i^{H^{(n)}}$ in $\mathscr{E}nd(H)$ is $\mathscr{I}_i^H$.

(ii) Immediate from (i) and, when $i > 0$, from (3.7.7).

(iii) Immediate from (i), since for $n \geq i$, $\mathscr{I}_i^H/\mathscr{I}_n^H$ identifies with the image of $\mathscr{I}_i^H$ in $\mathscr{I}_i^{H^{(n)}}$.

(iv) Immediate from (ii), since for $n \geq i$, $\mathscr{A}_i^H/\mathscr{A}_n^H$ identifies with the image of $\mathscr{A}_i^H$ in $\mathscr{A}_i^{H^{(n)}}$. $\qquad\square$

As a consequence of the proposition and of our earlier calculation of the the successive quotients of the $\mathscr{I}_\bullet^{H^{(n)}}$-filtration (3.7.13), we now obtain the successive quotients of the $\mathscr{I}_\bullet^H$-filtration. For any i and any $n \geq i+1$, we have monomorphisms

$$(**) \qquad \mathscr{I}_i^H/\mathscr{I}_{i+1}^H \hookrightarrow \mathscr{I}_i^{H^{(n)}}/\mathscr{I}_{i+1}^{H^{(n)}} \hookrightarrow \begin{cases} \mathbb{O}, & i = 0; \\ \mathbb{G}_a, & i > 0; \end{cases}$$

plainly the composite is independent of the choice of n.

Corollary 4.4.6. *The diagram* $(**)$ *induces an identification of presheaves*

$$\mathscr{I}_i^H / \mathscr{I}_{i+1}^H \cong \begin{cases} \mathbb{O}^{\mathrm{Fr}_{p^h}}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ p^3-1,\dots; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Fix i. For any $n \geq i+1$, we have an exact sequence of presheaves

$$0 \longrightarrow \mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow \mathscr{I}_i^{H^{(n)}} \longrightarrow \mathscr{I}_i^{H^{(n)}} / \mathscr{I}_{i+1}^{H^{(n)}} \longrightarrow 0.$$

It follows from (3.7.13) that

- $(\mathscr{I}_{i+1}^{H^{(n)}})_{n \geq i+1}$ satisfies the Mittag-Leffler condition as a diagram of presheaves of abelian groups; and
- as n increases, $\mathscr{I}_i^{H^{(n)}} / \mathscr{I}_{i+1}^{H^{(n)}}$ is eventually constant of the asserted value.

Now take the limit over n and use (4.4.5). $\qquad\square$

In an entirely similar fashion, using (3.7.3) in place of (3.7.13), and using the Mittag-Leffler condition for not-necessarily-abelian groups, we obtain the successive quotients of the $\mathscr{A}_\bullet^H$-filtration.

Corollary 4.4.7. *We have an identification of presheaves*

$$\mathscr{A}_i^H / \mathscr{A}_{i+1}^H \cong \begin{cases} \mu_{p^h-1}, & i = 0; \\ \mathbb{G}_a^{\mathrm{Fr}_{p^h}}, & i = p-1,\ p^2-1,\ p^3-1,\dots; \\ 0, & \text{otherwise.} \end{cases} \qquad\square$$

In the rest of the section we shall study the following quotient groups, which appear in (4.4.5), and their relation to the $\mathscr{E}nd(H^{(n)})$'s and $\mathscr{A}ut(H^{(n)})$'s.

Definition 4.4.8. We define $\mathscr{E}_n^H$ to be the presheaf quotient ring $\mathscr{E}nd(H)/\mathscr{I}_n^H$, and $\mathscr{U}_n^H$ to be the subpresheaf of units in $\mathscr{E}_n^H$.

In other words, by (3.7.8), $\mathscr{U}_n^H \cong \mathscr{A}ut(H)/\mathscr{A}_n^H$.

Remark 4.4.9. By (4.4.6) and (4.4.7), $\mathscr{E}_n^H$ and $\mathscr{U}_n^H$ can be obtained from finitely many iterated extensions of finite étale groups. Hence both are finite étale over $\operatorname{Spec}\mathbb{F}_p$. In fact, it is easy to write down explicit representing schemes. To fix ideas, consider $\mathscr{E}_n^H$. For all $i \geq 0$, the exact sequence of presheaves

$$0 \longrightarrow \mathscr{I}_{i+1}^H \longrightarrow \mathscr{I}_i^H \xrightarrow{\ \mathrm{can}\ } \mathscr{I}_i^H / \mathscr{I}_{i+1}^H \longrightarrow 0$$

has representable cokernel. Hence the quotient map "can" admits a section in the category of *set-valued* presheaves. Hence $\mathscr{I}_i^H \simeq \mathscr{I}_{i+1}^H \times (\mathscr{I}_i^H / \mathscr{I}_{i+1}^H)$ as presheaves of sets. Now, the possible nontrivial values of $\mathscr{I}_i^H / \mathscr{I}_{i+1}^H$, namely $\mathbb{O}^{\mathrm{Fr}_{p^h}}$ and $\mathbb{G}_a^{\mathrm{Fr}_{p^h}}$, both have underlying scheme $\operatorname{Spec}\mathbb{F}_p[T]/(T^{p^h} - T)$. Hence, letting l denote the integer such that $p^l \leq n < p^{l+1}$, we deduce that $\mathscr{E}_n^H$ is representable by

$$(\sharp) \qquad \operatorname{Spec}\mathbb{F}_p[T_0,\dots,T_l]/(T_0^{p^h} - T_0,\dots,T_l^{p^h} - T_l).$$

We can even specify a natural representation: S-points of $(\sharp)$ are canonically identified with ordered $(l+1)$-tuples of elements $a \in \Gamma(S)$ satisfying $a^{p^h} = a$, and we can take the map from $\mathscr{E}_n^H$ to $(\sharp)$ specified on points by sending the class of $f(T)$ to the coefficients of $T, T^p, \dots, T^{p^l}$.

Similarly, $\mathscr{U}_n^H$ is representable by

$$\operatorname{Spec} \mathbb{F}_p[T_0, T_0^{-1}, T_1, \ldots, T_l]/(T_0^{p^h} - T_0, \ldots, T_l^{p^h} - T_l).$$

Remark 4.4.10. Let us digress for a moment to make a remark on the $\mathscr{U}_n^H$'s. Let D denote the central division algebra over $\mathbb{Q}_p$ of dimension h^2 and Hasse invariant $\frac{1}{h}$. Let $\mathcal{O}_D$ denote the maximal order in D. Then a classical theorem of Dieudonné [D, Théorème 3] and Lubin [Lub, 5.1.3] in the theory of formal group laws asserts that $\mathcal{O}_D \simeq \operatorname{End}_{\mathbb{F}_{p^h}}(H)$ as topological rings, where $\operatorname{End}_{\mathbb{F}_{p^h}}(H)$ has the $\mathscr{I}_\bullet^H(\mathbb{F}_{p^h})$-topology; precisely, one has $p^r \mathcal{O}_D \simeq \mathscr{I}_{p^{rh}-1}^H(\mathbb{F}_{p^h})$ for all $r \geq 0$. Hence $\mathcal{O}_D^\times \simeq \operatorname{Aut}_{\mathbb{F}_{p^h}}(H) \cong \varprojlim_n \mathscr{U}_n^H(\mathbb{F}_{p^h})$ as pro-finite groups.

The finite algebraic group $\mathscr{U}_n^H$ and the abstract finite group $\mathscr{U}_n^H(\mathbb{F}_{p^h})$ are closely related: indeed, the former is a *twist* over $\operatorname{Spec} \mathbb{F}_p$ of the latter. Precisely, for any abstract group G and ring A, write G_A for the corresponding constant group scheme over $\operatorname{Spec} A$. Then $\mathscr{U}_n^H$ is not constant over $\operatorname{Spec} \mathbb{F}_p$, but it becomes isomorphic to $\left(\mathscr{U}_n^H(\mathbb{F}_{p^h})\right)_{\mathbb{F}_{p^h}}$ after the base change $\operatorname{Spec} \mathbb{F}_{p^h} \to \operatorname{Spec} \mathbb{F}_p$, as we see very explicitly from (4.4.9).

Our work so far furnishes a number of immediate relations between the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s, and between the $\mathscr{U}_n^H$'s and the $\mathscr{A}ut(H^{(n)})$'s. To fix ideas, let us consider the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s. For example, for all $n \geq 1$, $\mathscr{E}_n^H$ is identified with the image of $\mathscr{E}nd(H)$ in $\mathscr{E}nd(H^{(n)})$. And by (4.4.5), the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s have the same limit, namely $\mathscr{E}nd(H)$, endowed with the same topology. Our final goal for the section will be to show that a yet stronger statement holds: namely, that the $\mathscr{E}_n^H$'s and the $\mathscr{E}nd(H^{(n)})$'s determine *isomorphic pro-objects*; and similarly for the $\mathscr{U}_n^H$'s and the $\mathscr{A}ut(H^{(n)})$'s.

Precisely, let "$\varprojlim_n$" $\mathscr{E}_n^H$ be the pro-ring scheme obtained from the diagram

$$\cdots \longrightarrow \mathscr{E}_3^H \longrightarrow \mathscr{E}_2^H \longrightarrow \mathscr{E}_1^H,$$

and "$\varprojlim_n$" $\mathscr{E}nd(H^{(n)})$ be the pro-ring scheme obtained from the diagram

$$\cdots \longrightarrow \mathscr{E}nd(H^{(3)}) \longrightarrow \mathscr{E}nd(H^{(2)}) \longrightarrow \mathscr{E}nd(H^{(1)}).$$

The natural inclusions $\mathscr{E}_n^H \hookrightarrow \mathscr{E}nd(H^{(n)})$ for $n \geq 1$ plainly determine a morphism of pro-objects

$$\alpha\colon \text{``}\varprojlim_n\text{''} \mathscr{E}_n^H \longrightarrow \text{``}\varprojlim_n\text{''} \mathscr{E}nd(H^{(n)}).$$

We shall show that α is an isomorphism by exhibiting an explicit inverse β. To define β, we must define $\beta_n\colon \text{``}\varprojlim_m\text{''} \mathscr{E}nd(H^{(m)}) \to \mathscr{E}_n^H$ for each $n \geq 1$. For this, let l be the integer such that $p^l \leq n < p^{l+1}$, and take any $m \geq p^{l+h}$. Consider the natural map

$(\flat)$ $$\mathscr{E}nd(H^{(m)}) \longrightarrow \mathscr{E}nd(H^{(n)})$$

induced by truncation. By (3.7.13), (4.4.6), and choice of m, the image of $(\flat)$ in $\mathscr{E}nd(H^{(n)})$ identifies with $\mathscr{E}_n^H$. Hence $(\flat)$ induces "$\varprojlim_m$" $\mathscr{E}nd(H^{(m)}) \to \mathscr{E}_n^H$, which we take as the desired β_n. It is clear that the β_n's are compatible as n varies, so that we obtain the desired β.

Analogously, we may form the pro-algebraic groups

$$\text{``}\varprojlim_n\text{''} \mathscr{U}_n^H \quad \text{and} \quad \text{``}\varprojlim_n\text{''} \mathscr{A}ut(H^{(n)}),$$

and we obtain morphisms

$$\text{``}\varprojlim_{n}\text{''}\,\mathscr{U}_n^H \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} \text{``}\varprojlim_{n}\text{''}\,\mathscr{A}ut(H^{(n)}).$$

Theorem 4.4.11. *The morphisms α and β (resp., α' and β') are inverse isomorphisms of pro-objects.*

Proof. Everything is elementary from what we've already said. $\qquad\square$

4.5. **Functoriality of quotient stacks.** In the next section we shall need to interpret (4.4.11) in terms of the classifying stacks $B(\mathscr{U}_n)$ and $B\big(\mathscr{A}ut(H^{(n)})\big)$. We shall now pause a moment to record the following fact for use then: let $\mathscr{C}$ be a site, and let $\mathscr{D}$ denote the category of paris (G, X), where X is a sheaf on $\mathscr{C}$ and G is a group sheaf on $\mathscr{C}$ acting on X (on the left, say). Then passing to the quotient stack defines a morphism (in the sense of bicategories; see §A.2) from $\mathscr{D}$ to the 2-category of stacks over $\mathscr{C}$.

Roughly, the essential observation is simply that stackification defines a bicategory morphism $\mathrm{Fib}(\mathscr{C}) \to \mathrm{St}(\mathscr{C})$. More to the point, note that we have a canonical 2-functor $\mathscr{D} \to \mathrm{Fib}(\mathscr{C})$ sending (G, X) to the presheaf of groupoids

$$G \times X \underset{a}{\overset{\mathrm{pr}_X}{\rightrightarrows}} X,$$

where a denotes the action map. Hence the result of composing with stackification is to send $(G, X) \mapsto G\backslash X$.

In particular, taking X to be the sheaf with constant value $\{*\}$, we see that $G \mapsto B(G)$ defines a morphism from group sheaves on $\mathscr{C}$ to stacks.

4.6. **The stratum of height h formal Lie groups II.** In this section we apply the work of the previous two sections to give another characterization of the stack $\mathscr{FLG}^h$ of formal Lie groups of height h, $h \geq 1$. Recall the algebraic groups $\mathscr{U}_n^H$, $n \geq 1$, of (4.4.8).

Theorem 4.6.1. $\mathscr{FLG}^h \approx \varprojlim_{n} B_{\mathrm{f\acute{e}t}}(\mathscr{U}_n^H).$

Proof. The proof just consists of stringing together some of our previous results. By §4.5 and (A.8.2), the isomorphism of pro-objects

$$\text{``}\varprojlim_{n}\text{''}\,\mathscr{A}ut(H^{(n)}) \overset{\sim}{\longrightarrow} \text{``}\varprojlim_{n}\text{''}\,\mathscr{U}_n^H$$

from (4.4.11) induces an equivalence of stacks

$$\varprojlim_{n} B_{\mathrm{f\acute{e}t}}\big(\mathscr{A}ut(H^{(n)})\big) \overset{\approx}{\longrightarrow} \varprojlim_{n} B_{\mathrm{f\acute{e}t}}(\mathscr{U}_n^H).$$

By (3.6.8), we have an equivalence $\mathscr{B}_n^h \approx B_{\mathrm{f\acute{e}t}}\big(\mathscr{A}ut(H^{(n)})\big)$ for $n \geq p^{h+1}$, plainly compatible with truncation on the $\mathscr{B}_n^h$ side and with the transition maps induced by $\text{``}\varprojlim_{n}\text{''}\,\mathscr{A}ut(H^{(n)})$ on the $B_{\mathrm{f\acute{e}t}}\big(\mathscr{A}ut(H^{(n)})\big)$ side. Now use (4.3.10). $\qquad\square$

Remark 4.6.2. One may consider the equivalences

$$B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)\big) \approx \mathscr{FLG}^h \approx \varprojlim_{n} B_{\mathrm{f\acute{e}t}}(\mathscr{U}_n^H)$$

combined from (4.3.8) and (4.6.1) to be a stack analog of the theorem $\mathscr{O}_D^\times \simeq$ $\mathrm{Aut}_{\mathbb{F}_q}(H)$ discussed in (4.4.10). Indeed, $\mathscr{U}_n^H$ becomes constant after the base change $\mathrm{Spec}\,\mathbb{F}_{p^h} \to \mathrm{Spec}\,\mathbb{F}_p$, and we obtain equivalences over $\mathbb{F}_{p^h}$

$$B_{\mathrm{fpqc}}\big(\mathscr{A}ut(H)_{\mathbb{F}_{p^h}}\big) \approx \varprojlim_n B\big((\mathscr{U}_n^H)_{\mathbb{F}_{p^h}}\big) \approx \varprojlim B(\mathscr{O}_D^\times/N),$$

where the limit on the right runs through the open normal subgroups N of $\mathscr{O}_D^\times$.

5. Valuative criteria

In this section we shall conduct a basic investigation of some properties of the stacks $\mathscr{FLG}$ and $\mathscr{B}_n$, $n \geq 1$, related to valuative criteria. As in previous sections, we work with the notion of height relative to fixed prime p.

Theorem 5.1. *$\mathscr{B}_n$ is universally closed over $\mathrm{Spec}\,\mathbb{Z}$, and for all $h \geq 1$ and $n \geq p^h$, $\mathscr{B}_n^{\geq h}$ is universally closed over $\mathrm{Spec}\,\mathbb{F}_p$.*

Proof. The proof is the same in all cases, so let's just consider $\mathscr{B}_n$ over $\mathrm{Spec}\,\mathbb{Z}$. We apply the valuative criterion in [LMB, 7.3]. Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Let X be an n-bud over K. Then X admits a coordinate, so we may assume X is given by a bud law

$$F(T_1, T_2) = T_1 + T_2 + \sum_{2 \leq i+j \leq n} a_{ij} T_1^i T_2^j, \qquad a_{ij} \in K.$$

For changes of coordinate of the form $f(T) = \lambda T$ for nonzero $\lambda \in K$, we obtain

$$f\big[F\big(f^{-1}(T_1), f^{-1}(T_2)\big)\big] = T_1 + T_2 + \sum_{2 \leq i+j \leq n} a_{ij} \lambda^{1-i-j} T_1^i T_2^j.$$

So, by taking λ of sufficiently negative valuation, we see that F is K-isomorphic to a bud law defined over $\mathscr{O}$. $\square$

Remark 5.2. $\mathscr{B}_n$ is not *proper* over $\mathbb{Z}$ because it is not separated. Indeed, let $\mathscr{O}$ be a valuation ring with fraction field K. Then the natural functor

$$\mathscr{B}_n(\mathscr{O}) \longrightarrow \mathscr{B}_n(K)$$

is faithful but not full. For example, for the additive n-bud $\mathbb{G}_a^{(n)}$ (1.7.6) we have

$$\mathrm{Aut}_{\mathscr{O}}(\mathbb{G}_a^{(n)}) \subsetneq \mathrm{Aut}_K(\mathbb{G}_a^{(n)}),$$

since the latter contains automorphisms of the form $f(T) = \lambda T$ for λ of nonzero valuation.

Similarly, $\mathscr{B}_n^{\geq h}$ is not separated over $\mathbb{F}_p$.

Example 5.3. The following may be taken as an exhibition of the non-separatedness of $\mathscr{B}_n$ and of $\mathscr{FLG}$. Let $\mathscr{O}$ be a DVR with uniformizing element π and residue field of positive characteristic. Then the group law $F(T_1, T_2) := T_1 + T_2 + \pi T_1 T_2$ determines a formal Lie group over $\mathrm{Spec}\,\mathscr{O}$. Let $f(T) := \pi T$. Then, over the generic point η, we have

$$f\big[F\big(f^{-1}(T_1), f^{-1}(T_2)\big)\big] = T_1 + T_2 + T_1 T_2.$$

Hence f specifies an isomorphism $\widehat{\mathbb{A}}_\eta^F \xrightarrow{\sim} \widehat{\mathbb{G}}_m$. But $\widehat{\mathbb{G}}_m$ is certainly not isomorphic to $\widehat{\mathbb{A}}^F$ over $\mathrm{Spec}\,\mathscr{O}$, since $\widehat{\mathbb{A}}_{\mathscr{O}}^F$ reduces to $\widehat{\mathbb{G}}_a$ at the closed point. Hence $\widehat{\mathbb{G}}_m$ admits nonisomorphic extensions from the generic point to $\mathrm{Spec}\,\mathscr{O}$.

The failure of $\mathscr{B}_n$ and of $\mathscr{B}_n^{\geq h}$ to be separated prevents one from concluding formally that the valuative criterion used in the proof of (5.1) holds for $\mathscr{FLG}$ and for $\mathscr{FLG}^{\geq h}$, respectively. Nevertheless, these stacks do satisfy a kind of "formal universal closedness", in the following sense.

Theorem 5.4. *Let $\mathscr{O}$ be a valuation ring with field of fractions K.*

(i) *If K has characteristic 0, then $\mathscr{FLG}(\mathscr{O}) \to \mathscr{FLG}(K)$ is essentially surjective.*

(ii) *If K has characteristic p and is* separably closed, *then $\mathscr{FLG}^{\geq h}(\mathscr{O}) \to \mathscr{FLG}^{\geq h}(K)$ is essentially surjective.*

Proof. (i) As is well-known, over a $\mathbb{Q}$-algebra, every formal group law is isomorphic to the additive law.

(ii) By Lazard's theorem [L, Théorème IV], formal group laws over separably closed fields of characteristic p are classified up to isomorphism by their height. Now use that group laws of every height are defined over $\mathbb{F}_p$, hence over $\mathscr{O}$. $\square$

Our remarks in (5.2) suggest that the failure of $\mathscr{B}_n^{\geq h}$ to be separated is tied to the additive n-bud, which has "height ∞". So it is natural to ask if the *stratum* $\mathscr{B}_n^h$ is separated. But the answer here is also negative: by (3.6.8), (3.7.1), and (3.7.4), $\mathscr{B}_n^h$ is the classifying stack of a group $\mathscr{A}ut(H^{(n)})$ which is positive dimensional and affine, so that $\mathscr{A}ut(H^{(n)})$ is not proper, so that $B\big(\mathscr{A}ut(H^{(n)})\big)$ is not separated [LMB, 7.8.1(2)]. There is, however, a positive result when we take the limit over n.

Theorem 5.5. *Let $\mathscr{O}$ be a valuation ring and K its field of fractions. Then $\mathscr{FLG}^h(\mathscr{O}) \to \mathscr{FLG}^h(K)$ is fully faithful for all $h \geq 1$.*

Proof. Of course, the assertion only has content when $\operatorname{char} K = p$, since otherwise $\mathscr{FLG}^h(\mathscr{O}) = \mathscr{FLG}^h(K) = \emptyset$. So assume $\operatorname{char} K = p$. By (4.6.1), $\mathscr{FLG}^h \approx \varprojlim_n B_{\text{fét}}(\mathscr{U}_n^H)$, where $\mathscr{U}_n^H$ is the finite étale group scheme over $\mathbb{F}_p$ of (4.4.8). In particular, $\mathscr{U}_n^H$ is proper. Hence $B_{\text{fét}}(\mathscr{U}_n^H)$ is a separated algebraic stack over $\mathbb{F}_p$ [LMB, 7.8.1(2)]. Hence $B(\mathscr{U}_n^H)(\mathscr{O}) \to B(\mathscr{U}_n^H)(K)$ is fully faithful. Now use that a limit of fully faithful maps is fully faithful (A.3.5). $\square$

Remark 5.6. As noted in the introduction, when $\mathscr{O}$ is a *discrete* valuation ring, (5.5) is a special case of de Jong's theorem that, when $\operatorname{char} K = p$, the base change functor

$$(*) \qquad \left\{ \begin{array}{c} p\text{-divisible groups and} \\ \text{homomorphisms over } \mathscr{O} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} p\text{-divisible groups and} \\ \text{homomorphisms over } K \end{array} \right\}$$

is fully faithful [dJ, 1.2]. (Tate proved that $(*)$ is fully faithful when $\operatorname{char} K = 0$ [T, Theorem 4].) Note that (5.5) only asserts bijections between Isom sets of objects, not Hom sets, as in de Jong's theorem. But it appears that the methods used to prove (5.5) extend to give bijections between Hom sets, provided one considers stacks of *categories*, not just stacks of groupoids.

A. APPENDIX

This appendix is devoted to some of the basic aspects of limits in 2-categorical contexts; or more precisely, to limits in *bicategories*. From the point of view of general bicategory theory, the applications we shall have in the main body are all of a rather simple sort, and it would doubtless require less labor on the whole to

treat them in a more ad hoc fashion as they arise. But we believe that bicategories afford a convenient and natural setting in which to understand many of the various notions at play, and we have chosen to take a little time to work through some of the foundations.

Almost everything we shall discuss is probably well-known, but for some of the material — notably that in §§A.6–A.9 — we have failed to find suitable references. We extend our apologies to those whose work we have overlooked.

We have chosen to systematically ignore certain foundational set-theoretic issues typically resolved through consideration of universes or of regular cardinals.

A.1. **Bicategories.** A bicategory is a categorification of an (ordinary) category. We shall decline to recall the precise definition; see, for example, [CWM, XII §6], [Bo, 7.7.1], or — the original reference — [Bé, §1] for complete details. Let us instead content ourselves with a few informal remarks. A bicategory consists of objects; morphisms between objects, called 1-cells; and morphisms between 1-cells with common source and target, called 2-cells. The generalization from categories to bicategories can be understood to a large extent in terms of the familiar generalization from sets to categories. In an ordinary category, morphisms $A \to B$ form a set $\mathrm{Hom}(A, B)$; whereas in a bicategory, 1-cells $A \to B$ and the 2-cells between them form a *category* $\mathrm{Hom}(A, B)$. In an ordinary category, composition is specified by a function $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$; whereas in a bicategory, composition is a *functor* $\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$. In an ordinary category, associativity is expressed as an equality of functions

$$(\dagger) \qquad \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \times \mathrm{Hom}(C, D) \rightrightarrows \mathrm{Hom}(A, D).$$

In a bicategory, the arrows in ($\dagger$) become functors between categories. But in category theory, it is typically unnatural to demand that two functors be equal on the nose. A better notion of sameness is that the functors be naturally isomorphic. So in a bicategory, the functors in ($\dagger$) agree up to specified natural isomorphism; that is, the isomorphism is specified as part of the data of the bicategory. Similarly, each object C in a bicategory is equipped with a distinguished 1-cell $\mathrm{id}_C \in \mathrm{ob}\,\mathrm{Hom}(C, C)$, but left and and right composition with id_C is the identity functor only up to respective specified "left identity" and "right identity" natural isomorphisms. Moreover, the associativity and identity isomorphisms are not taken to be arbitrary, but are themselves subject to natural coherence constraints. Philosophically, one may view the associativity and identity isomorphisms as "canonical": the coherence constraints ensure that all possible ways to use the associativity and identity isomorphisms to obtain a 2-cell between given 1-cells yield a common result.

Remark A.1.1. A bicategory with strict associativity and strict identity 1-cells is a 2-*category*. Many examples of honest 2-categories occur in practice, e.g. (Cat).

Remark A.1.2. From some points of view, it would be more natural to use the term "2-category" in place of "bicategory"; under such usage, one would then refer to the objects described in (A.1.1) as "strict" 2-categories. But we shall defer to the now well-established conventions of category theory and use "2-category" for the strict notion and "bicategory" for the more general, "up-to-isomorphism" notion.

Remark A.1.3. We shall often refer to ordinary categories as 1-categories. Just as a set may be regarded as a category in which all morphisms are identities, a

1-category may be regarded as a bicategory, and indeed 2-category, in which all 2-cells are identities.

A 1-cell $f\colon A \to B$ in a bicategory is *fully faithful* if for every object C, the natural functor

$$\mathrm{Hom}(C, A) \xrightarrow{f_*} \mathrm{Hom}(C, B)$$

is fully faithful. In the case of the bicategory (Cat), we recover the usual notion of a fully faithful functor between 1-categories.

A 1-cell $f\colon A \to B$ in a bicategory is an *equivalence* if there exists a 1-cell $g\colon B \to A$ such that $gf \simeq \mathrm{id}_A$ and $fg \simeq \mathrm{id}_B$. Equivalently, f is an equivalence $\iff$ for every object C, the natural functor

$$\mathrm{Hom}(C, A) \xrightarrow{f_*} \mathrm{Hom}(C, B)$$

is an equivalence of categories $\iff$ for every object C, the natural functor

$$\mathrm{Hom}(B, C) \xrightarrow{f^*} \mathrm{Hom}(A, C)$$

is an equivalence of categories.

An object A in a bicategory is *groupoidal* if for every object C, $\mathrm{Hom}(C, A)$ is a groupoid. In (Cat), the groupoidal objects are precisely the groupoids.

For any bicategory $\mathscr{C}$, one obtains a bicategory $\mathscr{C}^{\mathrm{opp}}$ in a natural way by reversing 1-cells. There are two other notions of "opposite": one may reverse just the 2-cells, or the 1- and 2-cells simultaneously. We shall have no use for these latter two.

A.2. Morphisms of bicategories.

Part of the philosophy of category theory is that morphisms between mathematical objects are just as important as, if not more than, the objects themselves. So we shall now spend a few words on morphisms between bicategories. As in the previous section, we shall conduct the discussion at an informal level and leave precise details to the references. Let $\mathscr{I}$ and $\mathscr{C}$ and be bicategories.

In rough form, a *morphism*, or *diagram*, $F\colon \mathscr{I} \to \mathscr{C}$ consists of

- an assignment on objects $\mathrm{ob}\,\mathscr{I} \to \mathrm{ob}\,\mathscr{C}$; and
- for every pair $i, j \in \mathrm{ob}\,\mathscr{I}$, a functor $\mathrm{Hom}_{\mathscr{I}}(i, j) \to \mathrm{Hom}_{\mathscr{C}}(Fi, Fj)$,

compatible with identity 1-cells and composition of 1-cells up to specified coherent invertible 2-cells. We shall decline to make "coherent" more precise; see [Bé, 4.1] for the full definition.

Remark A.2.1. Our notion of morphism is often called "homomorphism" in the literature; see e.g. [Bé, 4.2]. More generally, one may consider a notion in which assignment of 1-cells is compatible with identities and composition only up to not-necessarily-invertible coherent 2-cells: it is this that many authors call "morphism", whereas we would call it *lax morphism*. The lax morphisms occupy an important place in the theory, but we will not have occasion to consider them further.

Example A.2.2 (Corepresentable and representable morphisms)**.** For each $C \in \mathrm{ob}\,\mathscr{C}$, $h^C := \mathrm{Hom}_{\mathscr{C}}(C, -)$ defines a morphism

$$\mathscr{C} \longrightarrow (\mathrm{Cat})$$
$$D \longmapsto \mathrm{Hom}_{\mathscr{C}}(C, D)$$

in a natural way. We call h^C the *morphism corepresented by C*. Similarly, $h_C :=$ Hom$(-, C)$ defines a morphism $\mathscr{C}^{\mathrm{opp}} \to (\mathrm{Cat})$; h_C is the *morphism represented by C*.

Remark A.2.3. When $\mathscr{I}$ is a 1-category, a bicategory morphism $\mathscr{I} \to \mathscr{C}$ is essentially a pseudofunctor as defined in [SGA1, VI §8] (strictly speaking, [SGA1] treats the case $\mathscr{C} = (\mathrm{Cat})$, but the modifications needed for general $\mathscr{C}$ are slight). In the main body, we shall only encounter morphisms for which $\mathscr{I}$ is a 1-category, so let us give a more explicit description in this case. Such a morphism $F \colon \mathscr{I} \to \mathscr{C}$ consists of the data

- for every object i in $\mathscr{I}$, an object Fi in $\mathscr{C}$;
- for every morphism $\mu \colon i \to j$ in $\mathscr{I}$, a 1-cell $F\mu \colon Fi \to Fj$ in $\mathscr{C}$;
- for every $i \in \mathrm{ob}\,\mathscr{I}$, an invertible 2-cell $\chi_i \colon F\mathrm{id}_i \xrightarrow{\sim} \mathrm{id}_{Fi}$ in $\mathscr{C}$; and
- for every composition $i \xrightarrow{\mu} j \xrightarrow{\nu} k$ in $\mathscr{I}$, an invertible 2-cell

$$\chi_{\nu,\mu} \colon \mathscr{F}(\nu\mu) \xrightarrow{\sim} (\mathscr{F}\nu)(\mathscr{F}\mu)$$

in $\mathscr{C}$.

These data are subject to natural coherence conditions expressing left and right identity and associativity constraints on the χ's. We shall decline to write out the constraints precisely.

In practice, the χ's typically arise as canonical isomorphisms. So, to simplify notation, we often just write same symbol "can" in place of the various χ's.

When $\mathscr{C}$ is a 2-category and one wishes to construct a morphism into $\mathscr{C}$, it can often be arranged that $F\mathrm{id}_i = \mathrm{id}_{Fi}$ and χ_i is the identity 2-cell for each $i \in \mathrm{ob}\,\mathscr{I}$.

Remark A.2.4. Morphisms of bicategories carry equivalences of objects to equivalences. A morphism of bicategories is itself called an *equivalence* if it is surjective on equivalence classes of objects and induces equivalences on all Hom categories.

Remark A.2.5. As in (A.1.2), there is a reasonable case for calling a morphism of bicategories a "2-functor". But we shall again defer to standard usage in category theory and reserve "2-functor" for a morphism between 2-categories which is strictly compatible with composition and with identities.

Just as functors between 1-categories admit natural transformations between them, so do morphisms between bicategories admit arrows, which we call just *transformations*, between them. In rough form, if F, $G \colon \mathscr{I} \to \mathscr{C}$ are bicategory morphisms, then a transformation $\alpha \colon F \to G$ consists of a 1-cell $\alpha_i \colon Fi \to Gi$ in $\mathscr{C}$ for each $i \in \mathrm{ob}\,\mathscr{I}$, compatible with the definitions of F and G on 1-cells up to specified coherent invertible 2-cells.

Remark A.2.6. We would obtain the notion of *lax transformation* by requiring compatibility only up to not-necessarily-invertible coherent 2-cells. Some authors use "transformation" for our notion of lax transformation, and "strong transformation" for our notion of transformation. As for lax morphisms between bicategories, we shall not have occasion to consider lax transformations further.

There is one further notion of arrow to consider, new to the setting of bicategories: if α, $\beta \colon F \to G$ are transformations, then a *modification* $\xi \colon \alpha \to \beta$ consists

of a (not-necessarily-invertible) 2-cell

$$Fi \overset{\alpha_i}{\underset{\beta_i}{\Downarrow \xi_i}} Gi$$

in $\mathscr{C}$ for each $i \in \mathrm{ob}\,\mathscr{I}$, subject to a natural coherence condition.

Just as functors from one category to another and the natural transformations between these functors are naturally organized into a category, the bicategory morphisms, transformations, and modifications are naturally organized as the respective objects, 1-cells, and 2-cells of a bicategory $\mathrm{Hom}(\mathscr{I},\mathscr{C})$. When $\mathscr{C}$ is a 2-category, so is $\mathrm{Hom}(\mathscr{I},\mathscr{C})$.

Remark A.2.7. Let F, $G\colon \mathscr{I} \to \mathscr{C}$ be morphisms. When $\mathscr{I}$ and $\mathscr{C}$ are 1-categories, it is a familiar fact that a natural transformation $F \to G$ is an isomorphism exactly when it is an objectwise isomorphism, i.e. when $Fi \to Gi$ is an isomorphism for all $i \in \mathrm{ob}\,\mathscr{I}$. Analogously, when $\mathscr{I}$ and $\mathscr{C}$ are arbitrary bicategories, a transformation $F \to G$ is an equivalence exactly when it is an objectwise equivalence.

Remark A.2.8 (Yoneda's lemma). Yoneda's lemma admits the following formulation in the setting of bicategories: for every object C in $\mathscr{C}$ and bicategory morphism $F\colon \mathscr{C}^{\mathrm{opp}} \to (\mathrm{Cat})$, the natural functor $\mathrm{Hom}_{\mathrm{Hom}(\mathscr{C},(\mathrm{Cat}))}(h_C, F) \to FC$ is an equivalence of categories. The analogous statement holds for h^C.

Remark A.2.9 (Diagonal morphism). There is a canonical morphism of bicategories $\mathscr{C} \to \mathscr{C}^{\mathscr{I}} := \mathrm{Hom}(\mathscr{I},\mathscr{C})$, called the *diagonal* and denoted $\Delta_{\mathscr{I},\mathscr{C}}$ or often just Δ, which we now describe.

As a warm-up, let us review the diagonal in the setting of 1-categories $\mathscr{I}$ and $\mathscr{C}$; see e.g. [CWM, III §3]. The diagonal is defined on each object C in $\mathscr{C}$ by taking ΔC to be the functor $\mathscr{I} \to \mathscr{C}$ with constant value C on objects and constant value id_C on morphisms. The diagonal is defined on morphisms in the evident way.

When $\mathscr{I}$ and $\mathscr{C}$ are bicategories, the diagonal is defined quite analogously. For each $C \in \mathrm{ob}\,\mathscr{C}$, there is a natural morphism $\mathscr{I} \to \mathscr{C}$ sending all objects to C, all 1-cells to id_C, and all 2-cells to $\mathrm{id}_{\mathrm{id}_C}$; this defines Δ on objects. There is then an evident way to define Δ on 1-cells and 2-cells in $\mathscr{C}$, and Δ admits a natural structure of bicategory morphism $\mathscr{C} \to \mathrm{Hom}(\mathscr{I},\mathscr{C})$.

A.3. **Limits in bicategories.** Let $\mathscr{C}$ be a 1-category. Informally, the limit of a diagram in $\mathscr{C}$ is an object L in $\mathscr{C}$ with the property that, for each object C in $\mathscr{C}$, to give a morphism from C to L is to give a family of morphisms from C, one to each object in the diagram, compatible with the transition morphisms in the diagram.

More precisely, let us say that the diagram in $\mathscr{C}$ is given by an "index" category $\mathscr{I}$ and a functor $F\colon \mathscr{I} \to \mathscr{C}$. Then a limit of F is a pair (L,α), where $L \in \mathrm{ob}\,\mathscr{C}$ and α is a natural transformation $\Delta L \to F$ (A.2.9), with the property that, for each object C in $\mathscr{C}$, the composition

$$\mathrm{Hom}_{\mathscr{C}}(C, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I},\mathscr{C})}(\Delta C, \Delta L) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I},\mathscr{C})}(\Delta C, F)$$

is a bijection of sets. As is standard, we write $\varprojlim F$ for L and typically omit α from the notation. By Yoneda's lemma, the pair $(\varprojlim F, \alpha)$ is well-defined up to canonical isomorphism.

The generalization to bicategories is straightforward. Let $F \colon \mathscr{I} \to \mathscr{C}$ be a morphism of bicategories. Again, recall the diagonal Δ of (A.2.9).

Definition A.3.1. A *limit* of F is a pair (L, α), where $L \in \mathrm{ob}\,\mathscr{C}$ and α is a transformation $\Delta L \to F$, with the property that, for each object C in $\mathscr{C}$, the composition of functors

$$\mathrm{Hom}_{\mathscr{C}}(C, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, \mathscr{C})}(\Delta C, \Delta L) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, \mathscr{C})}(\Delta C, F)$$

is an equivalence of 1-categories.

As for limits in 1-categories, we write $\varprojlim F$ for L and typically omit α from the notation. It is sometimes convenient to denote the limit more informally by $\varprojlim_{i \in \mathscr{I}} Fi$ or by $\varprojlim_i Fi$. By Yoneda's lemma, the pair $(\varprojlim F, \alpha)$ is well-defined up to equivalence.

Remark A.3.2. Of course, if $(\varprojlim F, \alpha)$ is a limit of F, then we obtain a "projection" 1-cell $\mathrm{pr}_i := \alpha_i \colon \varprojlim F \to Fi$ for each $i \in \mathrm{ob}\,\mathscr{I}$. As i varies, the pr_i's are compatible with the $F\mu$'s, $\mu \in \mathrm{mor}\,\mathscr{I}$, in the precisely the sense that α is a transformation $\Delta \varprojlim F \to F$.

Remark A.3.3. What we have called "limit" is usually called "bilimit" in the categorical literature. Consider the case that $\mathscr{I}$ and $\mathscr{C}$ are 2-categories and F is a 2-functor. Then, forgetting structure, F provides a functor between the underlying 1-categories. But the limit of F in the 1-categorical sense and the limit of F in the sense of (A.3.1) need not agree, even when $\mathscr{I}$ is a 1-category; see e.g. (A.4.2). Traditionally, "bilimit" has been used for limits in the sense of (A.3.1), whereas "limit" has been reserved for the 1-categorical notion. But in the setting of bicategories, (A.3.1) is the more fundamental notion. So we feel it deserves the plainer terminology, and it is that we shall call "limit". When we need to distinguish the 1-categorical notion, we shall refer to it as "1-limit" and denote it $1\text{-}\varprojlim F$.

Note that when $\mathscr{I}$ is a 2-category and $\mathscr{C}$ is a 1-category, the notions of limit of F and of 1-limit of F are the same. In particular, (A.3.1) recovers the usual notion of limit in the case of a functor between 1-categories.

Remark A.3.4. Of course, the limit of a given $F \colon \mathscr{I} \to \mathscr{C}$ may or may not exist. At the other extreme, suppose that $\mathscr{C}$ admits all $\mathscr{I}$-indexed limits, that is, that every morphism $\mathscr{I} \to \mathscr{C}$ admits a limit. Then, choosing a limit $(\varprojlim F, \alpha)$ for each $F \colon \mathscr{I} \to \mathscr{C}$, and using the universal property of a limit, it is straightforward (though tedious) to verify that passing to the limit admits a structure of bicategory morphism $\mathrm{Hom}(\mathscr{I}, \mathscr{C}) \to \mathscr{C}$, right adjoint to Δ (in the sense of bicategories).

Remark A.3.5. Let $G \colon \mathscr{I} \to \mathscr{C}$ be another morphism and $\alpha \colon F \to G$ a transformation. It follows immediately from the definitions that if α is fully faithful as a 1-cell in $\mathrm{Hom}(\mathscr{I}, \mathscr{C})$ and $\varprojlim F$ and $\varprojlim G$ exist, then the natural induced 1-cell $\varprojlim F \to \varprojlim G$ (well-defined up to isomorphism) in $\mathscr{C}$ is fully faithful. Moreover, there is the following easy criterion for α to be fully faithful, often satisfied in practice:

$$\alpha_i \text{ is fully faithful in } \mathscr{C} \text{ for all } i \in \mathrm{ob}\,\mathscr{I} \implies \alpha \text{ is fully faithful.}$$

Remark A.3.6. It follows immediately from the definitions that if F is a groupoidal object in $\mathrm{Hom}(\mathscr{I}, \mathscr{C})$ and $\varprojlim F$ exists, then $\varprojlim F$ is groupoidal in $\mathscr{C}$. Moreover,

there is the following easy criterion for F to be groupoidal, often satisfied in practice:

$$Fi \text{ is groupoidal in } \mathscr{C} \text{ for all } i \in \mathrm{ob}\,\mathscr{I} \implies F \text{ is groupoidal.}$$

Remark A.3.7. For later use, consider the situation of an object C in $\mathscr{C}$ and transformations $\alpha,\ \beta\colon \Delta C \to F$. For simplicity, assume that $\mathscr{I}$ is a 1-category; this will be the case of interest to us in applications. Then it is easy to verify that the assignment $i \mapsto \mathrm{Hom}_{\mathscr{C}}(\alpha_i, \beta_i)$ defines a functor $\mathscr{I} \to (\mathrm{Sets})$ in a natural way. Moreover, it is then immediate from the definition of modification that the limit $\varprojlim_{i\in\mathscr{I}} \mathrm{Hom}_{\mathscr{C}}(\alpha_i, \beta_i)$ of this functor is canonically identified with the set of modifications $\alpha \to \beta$. When $\mathscr{I}$ is an arbitrary bicategory, one still obtains a morphism $\mathscr{I} \to (\mathrm{Sets})$ whose limit is the set of modifications $\alpha \to \beta$, but the needed verifications require more work.

Remark A.3.8. More generally, Simpson [Si] has defined limits in n-categories and begun a study of them.

Remark A.3.9. More generally, there is a notion of *weighted limit* in a bicategory. See (A.4.5) for brief remarks and a reference.

A.4. **Limits of categories.** In this section we consider the important special case of limits in (Cat).

As motivation, let us begin with a brief review of limits in (Sets). Let $\mathscr{I}$ be a small 1-category and $F\colon \mathscr{I} \to (\mathrm{Sets})$ a functor. Of course, it is well-known that the limit of F exists. But even if we didn't know this, we could discover the limit in the following simple way. Let $*$ denote a singleton set, so that

$$S \cong \mathrm{Hom}_{(\mathrm{Sets})}(*, S)$$

for any set S. If $\varprojlim F$ exists, then the displayed formula and the universal property of the limit would force

$$\varprojlim F \cong \mathrm{Hom}_{(\mathrm{Sets})}(*, \varprojlim F) \cong \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, (\mathrm{Sets}))}(\Delta*, F).$$

On the other hand, we may take the right-hand side of this last display as a *definition* of the limit: one verifies directly that there is a canonical functorial bijection of sets

$$\mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, (\mathrm{Sets}))}(\Delta S, F) \cong \mathrm{Hom}_{(\mathrm{Sets})}\big(S, \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, (\mathrm{Sets}))}(\Delta*, F)\big)$$

for any set S; this is just the version for covariant functors of the familiar adjunction between forming the constant presheaf and taking global sections. Hence $\mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, (\mathrm{Sets}))}(\Delta*, F)$ has the universal property of the limit.

Everything generalizes to limits in (Cat) in a straightforward way. Let $F\colon \mathscr{I} \to (\mathrm{Cat})$ be a morphism of bicategories. We now regard $*$ as a category with a single object and single (identity) morphism. Then for any category C, there is a canonical isomorphism (not just equivalence) of categories

$$C \cong \mathrm{Hom}_{(\mathrm{Cat})}(*, C).$$

Hence, just as we saw for limits in (Sets), provided $\varprojlim F$ exists, we are forced to compute it as

$$(\dagger) \qquad\qquad \varprojlim F \cong \mathrm{Hom}_{(\mathrm{Cat})}(*, \varprojlim F) \approx \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I}, (\mathrm{Cat}))}(\Delta*, F).$$

On the other hand, one again finds that the right-hand side of (†) serves to *define* the limit: there is a canonical functorial isomorphism (not just equivalence) of categories

$$\operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},(\mathrm{Cat}))}(\Delta C, F) \cong \operatorname{Hom}_{(\mathrm{Cat})}\big(C, \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},(\mathrm{Cat}))}(\Delta *, F)\big)$$

for any category C. Hence $\operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},(\mathrm{Cat}))}(\Delta *, F)$ has the universal property of the limit.

Remark A.4.1. For each $i \in \operatorname{ob}\mathscr{I}$, the projection functor

$$\varprojlim F = \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},(\mathrm{Cat}))}(\Delta *, F) \xrightarrow{\operatorname{pr}_i} Fi$$

has a simple description. Indeed, let $X \colon \Delta * \to F$ be a transformation. Then $\operatorname{pr}_i X$ is just the object of Fi identified with $X_i \colon * \to Fi$ via the canonical isomorphism $Fi \cong \operatorname{Hom}(*, Fi)$.

Example A.4.2 (Fibered product of categories). Let

$$
\begin{array}{ccc}
& & D \\
& & \downarrow g \\
C & \xrightarrow{f} & E
\end{array}
$$

be a diagram of categories and functors, say given by an (honest) functor

$$
\begin{array}{ccc}
& \bullet & \\
& \downarrow & \\
\bullet & \longrightarrow & \bullet
\end{array}
\qquad \xrightarrow{F} \qquad (\mathrm{Cat}),
$$

where we regard the source of F as a 1-category $\mathscr{I}$. Then $\operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},(\mathrm{Cat}))}(\Delta *, F)$ is isomorphic to the category whose

- objects are tuples (X, Y, Z, φ, ψ), where

$$X \in \operatorname{ob}C, \quad Y \in \operatorname{ob}D, \quad Z \in \operatorname{ob}E,$$

and

$$\varphi \colon fX \xrightarrow{\sim} Z \quad \text{and} \quad \psi \colon gY \xrightarrow{\sim} Z$$

are isomorphisms in E; and

- morphisms $(X, Y, Z, \varphi, \psi) \to (X', Y', Z', \varphi', \psi')$ are triples (α, β, γ), where

$$\alpha \colon X \longrightarrow X', \quad \beta \colon Y \longrightarrow Y', \quad \text{and} \quad \gamma \colon Z \longrightarrow Z'$$

are morphisms in C, D, and E, respectively, making the evident diagrams commute.

We take the fibered product $C \times_E D$ to be the category so defined.

It is common to find $C \times_E D$ defined instead as the category of triples (X, Y, φ), where $X \in \operatorname{ob}C$, $Y \in \operatorname{ob}D$, and $\varphi \colon fX \xrightarrow{\sim} gY$ is an isomorphism in E. One readily checks that this category is equivalent, though not in general isomorphic, to the fibered product as we've defined it. In practice, we will only ever be interested in categories up to equivalence, and we will freely use either definition.

On the other hand, since F is an honest functor, we may speak of its 1-limit (A.3.3), that is, the 1-categorical fibered product: this is the category of pairs $X \in \operatorname{ob}C$ and $Y \in \operatorname{ob}D$ for which $fX = gY$. It is easy to write down simple choices of C, D, and E for which $\varprojlim F$ and 1-$\varprojlim F$ are not equivalent.

Example A.4.3 (Limit indexed on a 1-category). More generally, let $\mathscr{I}$ be a 1-category and $F\colon \mathscr{I} \to$ (Cat) a morphism of bicategories. Then, by (†), we may take as $\varprojlim F$ the category of families $(X_i, \varphi_\mu)_{i\in\mathrm{ob}\,\mathscr{I},\mu\in\mathrm{mor}\,\mathscr{I}}$, where

- for each object i in $\mathscr{I}$, X_i is an object in Fi, and
- for each morphism $\mu\colon i \to j$ in $\mathscr{I}$, φ_μ is an isomorphism $(F\mu)X_i \xrightarrow{\sim} X_j$ in Fj,

subject to the cocycle condition

- for every composition $i \xrightarrow{\mu} j \xrightarrow{\nu} k$ in $\mathscr{I}$, the diagram

$$
\begin{array}{ccc}
\big(\mathscr{F}(\nu\mu)\big)X_i & \xrightarrow[\sim]{\varphi_{\nu\mu}} & X_k \\
{\scriptstyle\mathrm{can}}\Big\downarrow{\scriptstyle\sim} & & {\scriptstyle\sim}\Big\uparrow{\scriptstyle\varphi_\nu} \\
(\mathscr{F}\nu)(\mathscr{F}\mu)X_i & \xrightarrow[\sim]{(\mathscr{F}\nu)\varphi_\mu} & (\mathscr{F}\nu)X_j
\end{array}
$$

commutes in Fk.

A morphism $(X_i, \varphi_\mu) \to (X_i', \varphi_\mu')$ is a family $(\alpha_i)_{i\in\mathrm{ob}\,\mathscr{I}}$, with $\alpha_i\colon X_i \to X_i'$ a morphism in Fi for each i, compatible with the $F\mu$'s and φ_μ's.

The reader may have noticed that, for fibered products in (A.4.2), the families we took as objects in $\varprojlim F$ did not include any morphisms φ indexed by an identity morphism in $\mathscr{I}$. It is an easy exercise to verify that, under the present description of $\varprojlim F$ in the case of fibered products, any φ indexed by an identity morphism must itself be an identity morphism. So the present description of $\varprojlim F$ agrees with that in (A.4.2) up to isomorphism (not just equivalence).

Remark A.4.4. It is a familiar fact in 1-category theory that limits in arbitrary categories can be characterized in terms of limits in (Sets). Indeed, let $F\colon \mathscr{I} \to \mathscr{C}$ be a functor between 1-categories. For each $C \in \mathrm{ob}\,\mathscr{C}$, let $h^C := \mathrm{Hom}_\mathscr{C}(C,-)$ be the functor $\mathscr{C} \to$ (Sets) corepresented by C. Then the universal property of the limit of F can be expressed as an isomorphism

$$(\ddagger) \qquad\qquad \mathrm{Hom}_\mathscr{C}(C, \varprojlim F) \simeq \varprojlim (h^C \circ F)$$

functorial in $C \in \mathrm{ob}\,\mathscr{C}$; or, more informally,

$$\mathrm{Hom}_\mathscr{C}\left(C, \varprojlim_{i\in\mathscr{I}} Fi\right) \simeq \varprojlim_{i\in\mathscr{I}} \mathrm{Hom}_\mathscr{C}(C, Fi).$$

The equivalence of the universal property formulated in (‡) with the universal property formulated in (A.3.1) is immediate, since there is a tautological identification

$$\varprojlim(h^C \circ F) \cong \mathrm{Hom}_{\mathrm{Hom}(\mathscr{I},\mathscr{C})}(\Delta C, F)$$

amounting to nothing more than the definition of natural transformation of functors.

Quite analogously, limits in bicategories can be characterized in terms of limits in (Cat): if $F\colon \mathscr{I} \to \mathscr{C}$ is a morphism of bicategories, then the universal property of the limit can be expressed as an equivalence of categories

$$\mathrm{Hom}_\mathscr{C}(C, \varprojlim F) \approx \varprojlim (h^C \circ F)$$

functorial (in the sense of bicategories) in $C \in \mathrm{ob}\,\mathscr{C}$; here $h^C = \mathrm{Hom}_\mathscr{C}(C,-)$ is the bicategory morphism $\mathscr{C} \to$ (Cat) corepresented by C (A.2.2). Indeed, this time the

definitions furnish a tautological isomorphism (not just equivalence) of categories

$$\varprojlim (h^C \circ F) = \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I}, (\mathrm{Cat}))}(\Delta*, h^C \circ F) \cong \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I}, \mathscr{C})}(\Delta C, F),$$

so that we recover the universal property of (A.3.1).

Remark A.4.5 (Weighted limits). One may take (A.4.4) as point of departure for the notion of *weighted limit* in a bicategory. As we saw, the limit of F is a representing object in $\mathscr{C}$ for the morphism

$$\mathscr{C}^{\mathrm{opp}} \longrightarrow (\mathrm{Cat})$$
$$C \longmapsto \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I}, (\mathrm{Cat}))}(\Delta*, h^C \circ F).$$

More generally, one may replace $\Delta*$ in the display by an arbitrary morphism $W : \mathscr{I} \to (\mathrm{Cat})$; a W-*weighted limit of* F is a representing object in $\mathscr{C}$ for the morphism $\mathscr{C}^{\mathrm{opp}} \to (\mathrm{Cat})$ so obtained. See [Lack, §6] for an informal introduction, and references therein for more details.

Remark A.4.6 (Morphisms in limits of categories). Let $F : \mathscr{I} \to (\mathrm{Cat})$ be a bicategory morphism, and let X and Y be objects in $\varprojlim F$. For simplicity, assume that $\mathscr{I}$ is a 1-category; this will be the case of interest to us in applications. Then by (A.3.7) and (A.4.1), we obtain a canonical identification, which we express in informal notation,

$$\operatorname{Hom}_{\varprojlim_{i \in \mathscr{I}} Fi}(X, Y) \cong \varprojlim_{i \in \mathscr{I}} \operatorname{Hom}_{Fi}(\mathrm{pr}_i X, \mathrm{pr}_i Y).$$

We have assumed that $\mathscr{I}$ is a 1-category only so that we may apply (A.3.7); one obtains the displayed formula for an arbitrary bicategory $\mathscr{I}$ as soon as one allows (A.3.7) for arbitrary $\mathscr{I}$.

Remark A.4.7. Limits of categories indexed by 1-categories were introduced in [SGA1, VI 5.5], and studied further in [Gi], via fibered categories; see also [SGA4$_2$, VI 6.10–.11]. Recall that if $\mathscr{I}$ is a 1-category, then there is a natural 2-functor

$$\operatorname{Hom}(\mathscr{I}, (\mathrm{Cat})) \longrightarrow \operatorname{Fib}(\mathscr{I}^{\mathrm{opp}})$$

[SGA1, VI §9] which is an equivalence of 2-categories. Let $F : \mathscr{I} \to (\mathrm{Cat})$ be a bicategory morphism, and denote the corresponding fibered category over $\mathscr{I}^{\mathrm{opp}}$ by $\mathscr{F}$. In particular, $\Delta* : \mathscr{I} \to (\mathrm{Cat})$ corresponds (up to isomorphism) to $\mathscr{I}^{\mathrm{opp}}$ itself, and we have

$$\varprojlim F = \operatorname{Hom}_{\operatorname{Hom}(\mathscr{I}, (\mathrm{Cat}))}(\Delta*, F) \approx \operatorname{Hom}_{\operatorname{Fib}(\mathscr{I}^{\mathrm{opp}})}(\mathscr{I}^{\mathrm{opp}}, \mathscr{F}).$$

The right-hand side is the notion of limit defined in [SGA1].

Limits of toposes indexed by 1-categories were first studied in [SGA4$_2$, VI §8]. In particular, (8.1.3.3) of loc. cit. discusses the general universal mapping property of the limit in the particular context of toposes.

A.5. **Limits in morphism bicategories.** It is a familiar fact in 1-category theory that if a category $\mathscr{D}$ contains all limits, then so does $\operatorname{Hom}(\mathscr{C}, \mathscr{D})$ for any category $\mathscr{C}$. More precisely, let $\mathscr{I}$ be a category and $F : \mathscr{I} \to \operatorname{Hom}(\mathscr{C}, \mathscr{D})$ a functor. Then for each $C \in \operatorname{ob} \mathscr{C}$, we obtain a diagram

$$F_C : \ \mathscr{I} \longrightarrow \mathscr{D}$$
$$i \longmapsto (Fi)(C).$$

If $\varprojlim F_C$ exists for all C, then the assignment $C \mapsto \varprojlim F_C$ defines a functor $L \colon \mathscr{C} \to \mathscr{D}$ in a natural way, and L is a naturally a limit of F. In more informal notation,

$$\left(\varprojlim_{i \in \mathscr{I}} Fi\right)(C) = \varprojlim_{i \in \mathscr{I}} (Fi)(C);$$

or in more informal words, limits are computed objectwise. In particular, suppose $\mathscr{D}$ contains all $\mathscr{I}$-indexed limits; that is, suppose $\varprojlim$ defines a functor $\operatorname{Hom}(\mathscr{I}, \mathscr{D}) \to \mathscr{D}$, right adjoint to $\Delta_{\mathscr{I}, \mathscr{D}}$ (A.2.9). Then $\operatorname{Hom}(\mathscr{C}, \mathscr{D})$ has all $\mathscr{I}$-indexed limits, and they are computed via the composition

$$(\dagger) \qquad \operatorname{Hom}\big(\mathscr{I}, \operatorname{Hom}(\mathscr{C}, \mathscr{D})\big) \cong \operatorname{Hom}\big(\mathscr{C}, \operatorname{Hom}(\mathscr{I}, \mathscr{D})\big) \xrightarrow{(\varprojlim)_*} \operatorname{Hom}(\mathscr{C}, \mathscr{D}).$$

As in previous sections, everything now generalizes to the situation of bicategories. Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{I}$ be bicategories and $F \colon \mathscr{I} \to \operatorname{Hom}(\mathscr{C}, \mathscr{D})$ a morphism. Then for each $C \in \operatorname{ob} \mathscr{C}$, we again get a morphism $F_C \colon \mathscr{I} \to \mathscr{D}$ sending $i \mapsto (Fi)(C)$. Suppose each F_C admits a limit, and fix a particular choice $\varprojlim F_C$ for each C. Of course, this time the assignment $C \mapsto \varprojlim F_C$ does not canonically define a morphism $\mathscr{C} \to \mathscr{D}$, since we only have equivalences

$$\operatorname{Hom}_{\mathscr{D}}\big(\varprojlim F_C, \varprojlim F_{C'}\big) \approx \varprojlim_{i} \operatorname{Hom}_{\mathscr{D}}\big(\varprojlim F_C, (Fi)(C')\big)$$

for varying C, $C' \in \operatorname{ob} \mathscr{C}$. Nevertheless, the right-hand side of the display allows us to choose an assignment of 1-cells

$$\big(C \xrightarrow{f} C'\big) \longmapsto \left(\varprojlim F_C \xrightarrow{\varprojlim f} \varprojlim F_{C'}\right)$$

as f runs through the 1-cells in $\mathscr{C}$. It is straightforward to verify that our choices determine a morphism $L \colon \mathscr{C} \to \mathscr{D}$ in a natural way, and moreover L is naturally a limit of F. As in the 1-categorical case, if $\mathscr{D}$ has all $\mathscr{I}$-indexed limits, so that $\varprojlim$ admits a structure of morphism $\operatorname{Hom}(\mathscr{I}, \mathscr{D}) \to \mathscr{D}$, then $\operatorname{Hom}(\mathscr{C}, \mathscr{D})$ has all $\mathscr{I}$-indexed limits; these last may be computed exactly as in $(\dagger)$, interpreted in the setting of bicategories.

A.6. **Limits of fibered categories and of stacks.** In this section we shall apply some of the general considerations in previous sections to the particular setting of fibered categories and stacks. Let $\mathscr{C}$ be a 1-category. Strictly speaking, we shall apply our previous work directly to $\operatorname{Hom}(\mathscr{C}^{\mathrm{opp}}, (\mathrm{Cat}))$, and then to $\operatorname{Fib}(\mathscr{C})$ via the equivalence of 2-categories

$$\operatorname{Hom}(\mathscr{C}^{\mathrm{opp}}, (\mathrm{Cat})) \longrightarrow \operatorname{Fib}(\mathscr{C})$$

we recalled earlier in (A.4.7).

Our first result is an immediate consequence of §§A.4 and A.5.

Theorem A.6.1. $\operatorname{Fib}(\mathscr{C})$ *contains all limits.* $\qquad \square$

Remark A.6.2. Quite explicitly, let $F \colon \mathscr{I} \to \operatorname{Fib}(\mathscr{C})$ be a morphism of bicategories, and let $(Fi)(C)$ denote the category fiber of Fi over each $C \in \operatorname{ob} \mathscr{C}$. Then $\varprojlim F$ is the fibered category whose fiber over each C is the category $\varprojlim_{i \in \mathscr{I}} (Fi)(C)$. In other words, limits of fibered categories are computed fiberwise.

Of course, (A.6.1) is a generalization of the familiar fact that for any category $\mathscr{C}$, the category of presheaves of sets on $\mathscr{C}$ contains all (small) limits.

In the case of a diagram of CFG's in $\mathrm{Fib}(\mathscr{C})$, (A.3.6) asserts that the limit will again be a CFG. So we deduce the following.

Corollary A.6.3. $\mathrm{CFG}(\mathscr{C})$ *contains all limits; they are computed exactly as limits in* $\mathrm{Fib}(\mathscr{C})$. $\qquad\square$

Let us now turn to stacks. We continue with our 1-category $\mathscr{C}$, and suppose it equipped with a Grothendieck topology; we shall not consider here Grothendieck topologies on arbitrary bicategories. It is a familiar fact in sheaf theory that a limit of sheaves (of sets, say), computed in the category of presheaves, is always again a sheaf. We shall now arrive at the analogous result for stacks.

Let $F\colon \mathscr{I} \to \mathrm{Fib}(\mathscr{C})$ be a morphism of bicategories; then $\varprojlim F$ exists by (A.6.1).

Theorem A.6.4. *Suppose* Fi *is a stack for each* $i \in \mathrm{ob}\,\mathscr{I}$. *Then* $\varprojlim F$ *is a stack. In particular,* $\mathrm{St}(\mathscr{C})$ *contains all limits.*

Proof. Let $C \in \mathrm{ob}\,\mathscr{C}$ and $U \to \underline{C}$ a covering sieve on C. We must show that the natural functor

$$(\dagger) \qquad\qquad \mathrm{Hom}_{\mathrm{Fib}(\mathscr{C})}(\underline{C}, \varprojlim F) \longrightarrow \mathrm{Hom}_{\mathrm{Fib}(\mathscr{C})}(U, \varprojlim F)$$

is an equivalence of categories. By (A.4.4), we may replace $(\dagger)$ with

$$(\ddagger) \qquad\qquad \varprojlim_{i \in \mathscr{I}} \mathrm{Hom}_{\mathrm{Fib}(\mathscr{C})}(\underline{C}, Fi) \longrightarrow \varprojlim_{i \in \mathscr{I}} \mathrm{Hom}_{\mathrm{Fib}(\mathscr{C})}(U, Fi).$$

Since Fi is a stack for each $i \in \mathrm{ob}\,\mathscr{I}$, each $\mathrm{Hom}(\underline{C}, Fi) \to \mathrm{Hom}(U, Fi)$ is an equivalence. Hence $\mathrm{Hom}_{\mathrm{Fib}(\mathscr{C})}(\underline{C}, F-)$ and $\mathrm{Hom}_{\mathrm{Fib}(C)}(U, F-)$ are equivalent objects in $\mathrm{Hom}(\mathscr{I}, (\mathrm{Cat}))$ (A.2.7). Hence $(\ddagger)$ is an equivalence. $\qquad\square$

A.7. **Limits of objects in limits of categories; group objects.** Let $F\colon \mathscr{I} \to$ (Cat) be a morphism of bicategories. By §A.4, there exists a limit category $\varprojlim F$. Let now $\mathscr{J}$ be a 1-category and $G\colon \mathscr{J} \to \varprojlim F$ a functor. In this section, we wish to describe the limit of G in $\varprojlim F$ in terms of limits in the categories Fi, $i \in \mathrm{ob}\,\mathscr{I}$, under certain natural assumptions on F and G.

We shall make use below of (A.4.6). So, as in (A.4.6), we shall make the simplifying assumption that $\mathscr{I}$ is a 1-category. Everything we shall do would work for an arbitrary bicategory $\mathscr{I}$, but we only need the 1-categorical case in applications. We'll leave the verifications needed for the general case to the reader.

Our assumptions on F and G are as follows. For each $i \in \mathrm{ob}\,\mathscr{I}$, let G_i denote the composite functor

$$\mathscr{J} \xrightarrow{\ G\ } \varprojlim F \xrightarrow{\ \mathrm{pr}_i\ } Fi.$$

We shall assume

- for all objects i in $\mathscr{I}$, G_i admits a limit in Fi; and
- for all morphisms $\mu\colon i \to i'$ in $\mathscr{I}$, the functor $F\mu\colon Fi \to Fi'$ sends limits for Gi to limits for Gi'.

For each $i \in \mathrm{ob}\,\mathscr{I}$, choose a limit $\varprojlim G_i$ of G_i in Fi. By assumption, for each $\mu\colon i \to i'$ in $\mathscr{I}$, the objects $(F\mu)(\varprojlim G_i)$ and $\varprojlim G_{i'}$ are canonically isomorphic in Fi'. Hence the $\varprojlim G_i$'s determine an object L in $\varprojlim F$ (A.4.3).

Proposition A.7.1. *The object* L *is naturally a limit of* G *in* $\varprojlim F$.

Proof. Let $X \in \mathrm{ob} \varprojlim F$. Then

$$\mathrm{Hom}_{\mathrm{Hom}(\mathscr{J}, \varprojlim F)}(\Delta X, G) \cong \mathrm{Hom}_{\varprojlim_{i \in \mathscr{I}} \mathrm{Hom}(\mathscr{J}, Fi)}(\Delta X, G) \qquad \text{(A.4.4)}$$

$$\cong \varprojlim_{i \in \mathscr{I}} \mathrm{Hom}_{\mathrm{Hom}(\mathscr{J}, Fi)}(\Delta \mathrm{pr}_i X, G_i) \quad \text{(A.4.6)}$$

$$\cong \varprojlim_{i \in \mathscr{I}} \mathrm{Hom}_{Fi}(\mathrm{pr}_i X, \varprojlim G_i)$$

$$\cong \mathrm{Hom}_{\varprojlim F}(X, L) \qquad \text{(A.4.6).}$$

Hence L has the universal property of the limit. $\qquad\qquad\qquad\qquad \square$

As an application, let us now consider group objects in $\varprojlim F$. Quite generally, for any category $\mathscr{C}$, write $\mathrm{Gp}(\mathscr{C})$ for the category of group objects and homomorphisms in $\mathscr{C}$. Let us note that there is some ambiguity as to what we really mean by this category: namely, the products of objects in $\mathscr{C}$ are only defined up to canonical isomorphism, so some care is needed in defining the objects of $\mathrm{Gp}(\mathscr{C})$ precisely. Nevertheless, it is clear that all reasonable notions of $\mathrm{Gp}(\mathscr{C})$ produce equivalent categories. So we shall regard the issue as one of pedantry and ignore it from now on. Similarly, we write $\mathrm{Ab}(\mathscr{C})$ for the full subcategory of $\mathrm{Gp}(\mathscr{C})$ of commutative group objects in $\mathscr{C}$.

We continue with a 1-category $\mathscr{I}$ and a morphism of bicategories $F \colon \mathscr{I} \to (\mathrm{Cat})$. Suppose that

- for all $i \in \mathrm{ob}\,\mathscr{I}$, Fi admits all finite (including empty) products; and
- for all $\mu \in \mathrm{mor}\,\mathscr{I}$, $F\mu$ preserves all finite (including empty) products.

Then each $F\mu \colon Fi \to Fi'$ induces $\mathrm{Gp}(Fi) \to \mathrm{Gp}(Fi')$, and we obtain a morphism of bicategories

$$\mathscr{I} \longrightarrow (\mathrm{Cat})$$
$$i \longmapsto \mathrm{Gp}(Fi).$$

Moreover, by (A.7.1), the projection $\varprojlim_{i'} Fi' \xrightarrow{\mathrm{pr}_i} Fi$ induces $\mathrm{Gp}(\varprojlim_{i'} Fi') \to \mathrm{Gp}(Fi)$ for all $i \in \mathrm{ob}\,\mathscr{I}$, and we obtain an arrow

$$(\sharp) \qquad\qquad \mathrm{Gp}\Big(\varprojlim_{i \in \mathscr{I}} Fi\Big) \longrightarrow \varprojlim_{i \in \mathscr{I}} \mathrm{Gp}(Fi).$$

Similarly, we obtain the category $\varprojlim_i \mathrm{Ab}(Fi)$ and an arrow

$$(\flat) \qquad\qquad \mathrm{Ab}\Big(\varprojlim_{i \in \mathscr{I}} Fi\Big) \longrightarrow \varprojlim_{i \in \mathscr{I}} \mathrm{Ab}(Fi).$$

Proposition A.7.2. *The arrows $(\sharp)$ and $(\flat)$ are equivalences of categories.*

Proof. Immediate from (A.4.6) and (A.7.1). $\qquad\qquad\qquad\qquad\qquad \square$

A.8. **Pro-objects and limits.** Let

- $\mathscr{C}$ be a 1-category;
- pro-$\mathscr{C}$ denote the category of pro-objects in $\mathscr{C}$;
- $\mathscr{D}$ be a bicategory; and
- $F \colon \mathscr{C} \to \mathscr{D}$ be a morphism of bicategories.

Our goal for the section is to show that if $\mathscr{D}$ contains all filtered limits, then F induces a morphism $\underleftarrow{F} \colon \text{pro-}\mathscr{C} \to \mathscr{D}$ in a "natural" — or more precisely, in a canonical-up-to-equivalence — way.

For convenience, let us begin by recalling some of the basic definitions involved. A 1-category $\mathscr{I}$ is *filtered* if

(1) $\mathscr{I}$ is nonempty;
(2) for every i_1, $i_2 \in \text{ob}\,\mathscr{I}$, there exist $i \in \text{ob}\,\mathscr{I}$ and morphisms

$$i < \begin{array}{c} \dashrightarrow i_1 \\ \dashrightarrow i_2; \end{array}$$

and
(3) for every diagram

$$i_1 \rightrightarrows i_2$$

in $\mathscr{I}$, there exist $i \in \text{ob}\,\mathscr{I}$ and an equalizing morphism

$$i \dashrightarrow i_1 \rightrightarrows i_2,$$

that is, the two compositions in this last diagram are equal.

For applications, our primary interest is in limits indexed on the filtered set $\mathbb{N}$ (e.g. $\varprojlim_n (n\text{-Inf})$ in §2.4, $\varprojlim_n \mathscr{G}_n$ in §2.5, and $\varprojlim_n \mathscr{B}_n$ in §2.6), where we regard $\mathbb{N}$ as the category

$$\cdots \longrightarrow 3 \longrightarrow 2 \longrightarrow 1,$$

with compositions and identities suppressed.

Remark A.8.1. Many authors use "filtered" in the sense dual to ours; see e.g. [SGA4$_1$, I 2.7] or [CWM, IX §1]. In our use of the terminology, one typically encounters filtered limits and cofiltered colimits.

A *pro-object in $\mathscr{C}$* [SGA4$_1$, I §8.10] is a functor $X \colon \mathscr{I} \to \mathscr{C}$ defined on some small filtered category $\mathscr{I}$. We typically write X_i for Xi, $i \in \text{ob}\,\mathscr{I}$, and "$\varprojlim$"$_{i \in \mathscr{I}} X_i$ for X. The pro-objects in $\mathscr{C}$ form a category pro-$\mathscr{C}$, with morphisms defined by

$$\text{Hom}_{\text{pro-}\mathscr{C}} \left(\text{``}\varprojlim_{i \in \mathscr{I}}\text{''} X_i, \text{``}\varprojlim_{j \in \mathscr{J}}\text{''} Y_j \right) := \varprojlim_{j \in \mathscr{J}} \varinjlim_{i \in \mathscr{I}} \text{Hom}_{\mathscr{C}}(X_i, Y_j).$$

The composition law in $\mathscr{C}$ induces a natural one in pro-$\mathscr{C}$, well-defined, as one readily checks, because the index categories are filtered.

For simplicity, we shall consider neither the notion of filtered nor that of pro-object in the setting of arbitrary bicategories.

Let us now explain the construction of the morphism $\underleftarrow{F} \colon \text{pro-}\mathscr{C} \to \mathscr{D}$. It is useful to first consider the case $\mathscr{D}$ is a 1-category. On objects, $\underleftarrow{F}$ sends the pro-object $X \colon \mathscr{I} \to \mathscr{C}$ to $\varprojlim(F \circ X)$; or, more informally,

$$\text{``}\varprojlim_{i \in \mathscr{I}}\text{''} X_i \overset{F}{\longmapsto} \varprojlim_{i \in \mathscr{I}} FX_i.$$

$\underleftarrow{F}$ is defined on morphisms in the evident way, using the universal property of the limit. It is then easy to verify, using that $\mathscr{I}$ is filtered, that $\underleftarrow{F}$ is an honest functor. Plainly, $\underleftarrow{F}$ is well-defined up to canonical isomorphism.

In the case of a general bicategory $\mathscr{D}$, essentially the same approach works, but a bit more care must be taken in the definition of $\underleftarrow{F}$ on morphisms. For each pro-object $X = \text{``}\underleftarrow{\lim}\text{''}_{i \in \mathscr{I}}\, X_i$ in $\mathscr{C}$, we define $\underleftarrow{F}X$ to be any choice of limit $\underleftarrow{\lim}_{i \in \mathscr{I}}\, FX_i$ in $\mathscr{D}$. For each morphism of pro-objects

$$f = (f_j)_{j \in \mathrm{ob}\,\mathscr{J}} \in \mathrm{Hom}_{\mathrm{pro}\text{-}\mathscr{C}}\left(\text{``}\underleftarrow{\lim}_{i \in \mathscr{I}}\text{''}\, X_i,\, \text{``}\underleftarrow{\lim}_{j \in \mathscr{J}}\text{''}\, Y_j \right) = \underleftarrow{\lim}_{j \in \mathscr{J}} \underrightarrow{\lim}_{i \in \mathscr{I}} \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j),$$

for each $j \in \mathrm{ob}\, J$, choose a representative $\widetilde{f}_j \colon X_{i_j} \to Y_j$ of f_j. Then we get a 1-cell $\underleftarrow{f_j} \colon \underleftarrow{\lim}_i FX_i \to FY_j$ in $\mathscr{D}$ by composing

$$\underleftarrow{\lim}_i FX_i \xrightarrow{\mathrm{pr}_{i_j}} FX_{i_j} \xrightarrow{F\widetilde{f}_j} FY_j \ .$$
$$\underbrace{\phantom{\underleftarrow{\lim}_i FX_i \xrightarrow{\mathrm{pr}_{i_j}} FX_{i_j}}}_{\underleftarrow{f_j}}$$

For each $\nu \colon j_1 \to j_2$ in $\mathscr{J}$, denote by ν_* the induced morphism $Y_{j_1} \to Y_{j_2}$. One then verifies, using that $\mathscr{I}$ is filtered, that the diagram

$$\begin{array}{ccc}
 & \underleftarrow{\lim}_i FX_i & \\
{\scriptstyle \underleftarrow{f_{j_1}}} \swarrow & & \searrow {\scriptstyle \underleftarrow{f_{j_2}}} \\
FY_{j_1} & \xrightarrow[F\nu_*]{} & FY_{j_2}
\end{array}$$

commutes up to canonical isomorphism. Hence, by the universal property of the limit (A.4.3, A.4.4), we obtain a 1-cell $\underleftarrow{\lim}_i FX_i \to \underleftarrow{\lim}_j FY_j$, which we define to be $\underleftarrow{F}f$.

Proposition A.8.2. *The assignments $X \mapsto \underleftarrow{F}X$ on objects and $f \mapsto \underleftarrow{F}f$ on morphisms admit a natural structure of bicategory morphism $\underleftarrow{F} \colon \mathrm{pro}\text{-}\mathscr{C} \to \mathscr{D}$.*

The proof is straightforward, and we shall leave the details to the reader. Of course, the notion of filtered category plays an essential role.

It is clear that $\underleftarrow{F}$ is well-defined up to equivalence.

Remark A.8.3. There is a canonical functor $\chi \colon \mathscr{C} \to \mathrm{pro}\text{-}\mathscr{C}$ sending each object C to the pro-object indexed on $*$ with value C. Moreover, one verifies easily that pro-$\mathscr{C}$ contains all small filtered limits. One may now distinguish pro-$\mathscr{C}$ and χ amongst the maps from $\mathscr{C}$ into 1-categories by the following universal mapping property: up to isomorphism of functors, to give a functor $\mathscr{C} \to \mathscr{D}$ into a 1-category $\mathscr{D}$ containing small filtered limits is to give a functor pro-$\mathscr{C} \to \mathscr{D}$ preserving small filtered limits. Analogously, pro-$\mathscr{C}$ and χ enjoy a universal position amongst the bicategories: up to equivalence of morphisms of bicategories, to give a morphism $\mathscr{C} \to \mathscr{D}$ into a bicategory $\mathscr{D}$ containing small filtered limits is to give a morphism pro-$\mathscr{C} \to \mathscr{D}$ preserving small filtered limits.

A.9. **Initiality and limits.** It is a standard fact in 1-category theory that limits are invariant under right composition with an initial functor. Precisely, recall [CWM, IX §3] that a functor $L \colon \mathscr{J} \to \mathscr{I}$ is *initial* if for all $i \in \mathrm{ob}\,\mathscr{I}$, the comma category $\mathscr{J}_{/i}$ (denoted $L \downarrow i$ in the notation of [CWM]) is nonempty and connected. If $L \colon \mathscr{J} \to \mathscr{I}$ is initial, $F \colon \mathscr{I} \to \mathscr{C}$ is any functor, and $\underleftarrow{\lim} FL$ exists, then one proves easily that $\underleftarrow{\lim} FL$ is canonically a limit of F [CWM, IX 3.1].

In this section we shall obtain a suitable generalization to the following situation:

- $\mathscr{C}$ is an arbitrary bicategory;
- for simplicity, $\mathscr{I}$ and $\mathscr{J}$ are 1-categories, and $L\colon \mathscr{J} \to \mathscr{I}$ is an initial functor between them; and
- F is an arbitrary bicategory morphism $\mathscr{I} \to \mathscr{C}$.

Let $C \in \operatorname{ob}\mathscr{C}$.

Proposition A.9.1. *The natural functor*

$$\operatorname{Hom}_{\operatorname{Hom}(\mathscr{I},\mathscr{C})}(\Delta_{\mathscr{I},\mathscr{C}}C, F) \longrightarrow \operatorname{Hom}_{\operatorname{Hom}(\mathscr{J},\mathscr{C})}(\Delta_{\mathscr{J},\mathscr{C}}C, FL)$$

induced by L is an equivalence of categories.

Sketch proof. The proof is tedious but straightforward, so we'll just give a sketch. Let Φ denote the functor in the statement of the proposition.

One sees that Φ is faithful directly from the definition of 2-cells in morphism bicategories and from the assumption that every comma category $\mathscr{J}_{/i}$ is nonempty.

To see that Φ is full, suppose there are 1-cells

in $\operatorname{Hom}(\mathscr{I}, \mathscr{C})$ and a 2-cell

in $\operatorname{Hom}(\mathscr{J}, \mathscr{C})$. We must find a 2-cell $\widetilde{\xi}\colon \alpha \to \beta$ mapping to ξ. For each $i \in \operatorname{ob}\mathscr{I}$, choose an object $(j, Lj \xrightarrow{\mu} i)$ in $\mathscr{J}_{/i}$. One then defines $\widetilde{\xi}_i$ in an obvious way using ξ_j and μ. One next shows that $\widetilde{\xi}_i$ is independent of the choice of object in $\mathscr{J}_{/i}$, first in the special case that two choices admit a morphism in $\mathscr{J}_{/i}$ between them, and then in general using the special case and that $\mathscr{J}_{/i}$ is connected. It then follows easily from independence of choices that the $\widetilde{\xi}_i$'s define a 2-cell $\widetilde{\xi}$ mapping to ξ.

To see that Φ is essentially surjective, let $\alpha\colon \Delta C \to FL$ be a 1-cell in $\operatorname{Hom}(\mathscr{J}, \mathscr{C})$. As usual, for each $i \in \operatorname{ob}\mathscr{I}$, choose $(j, Lj \xrightarrow{\mu} i)$ in $\mathscr{J}_{/i}$, and define $\widetilde{\alpha}$ to be the composite 1-cell

$$C \xrightarrow{\alpha_j} FLj \xrightarrow{F\mu} Fi \ .$$
$$\underbrace{\phantom{C \xrightarrow{\alpha_j} FLj \xrightarrow{F\mu} Fi}}_{\widetilde{\alpha}_i}$$

This time $\widetilde{\alpha}_i$ is not independent of the choice of j and μ. But using that $\mathscr{J}_{/i}$ is connected, one verifies that different choices produce canonically isomorphic $\widetilde{\alpha}_i$'s. So the $\widetilde{\alpha}_i$'s define $\widetilde{\alpha}\colon \Delta C \to F$ in a natural way, and moreover $\Phi\widetilde{\alpha} \cong \alpha$. $\qquad\square$

Corollary A.9.2. *F admits a limit $\iff$ FL does, and the limits are equivalent when they exist.* $\qquad\square$

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